An Argument for **P=NP**

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- blue = new material created in light of a particular commentator
- red = new material created in light of 2nd commentator (who claimed that we smuggle in exponentially growing hardware)
- purple = latest additions due to our own further research and reflection

*We're greatly indebted to Michael Zenzen for many valuable discussions about the **P**=?**NP** problem and physics (*simpliciter* and digital), and to Jim Fahey for discussions about such physics and mixed-mode dual-diamond operators in modal logic. The presentation of the core arguments herein to the Spring 2005 edition of Bringsjord's graduate seminar, *Logic & Artificial Intelligence*, and his guest lectures on **P**=?**NP** in the 2005 *Formal Foundations of Cognitive Science* graduate seminar, sparked a number of helpful objections and suggestions, for which we are grateful. We're indebted as well to two anonymous commentators for trenchant comments. Though the two arguments herein (the second of which seems to establish **P**=**NP**) are for weal or woe Bringsjord's, Taylor's astute objections catalyzed crucial refinements.

The Clay Mathematics Institute offers a $1 million prize for a solution to the **P**=?**NP** problem.[1] We look forward to receiving our award — but concede that the expected format of a solution is an *object-level* proof, not a meta-level argument like what we provide. On the other hand, certainly the winner needn't provide a *constructive* proof that **P**=**NP**.[2] Despite Gödel's recently discovered position on the matter,[3] the general consensus has certainly been that **P**≠**NP**, and many of those brave, contrarian (and, alas, often confused) souls who have endeavored to show **P**=**NP** have sought to take the beckoning route of exhibiting a polynomial-time algorithm for one or more of the 1000 or so currently catalogued **NP**-complete problems. This is an exceedingly taxing (and, at least hitherto, unproductive) direction to take, and we eschew it. We happily concede that constructive success would have many practical implications, but we are more interested in the fact of the matter than, say, whether many current cryptographic schemes can be compromised. Very well; let's proceed.

In logic and related fields we often speak about problems in purely abstract terms. For example, we may declare a problem to be Turing-solvable, without giving any thought whatsoever to the *embodiment* of a Turing machine able to carry out a solution.[4] So we may for instance say that the set $\mathcal{C}$ of composite numbers is Turing-decidable: that there exists some TM M such that, for every $n \in \mathcal{N} = \{0, 1, 2, \ldots\}$, with n given to M as input (suitably encoded on its tape), M produces (say) Y iff $n \in \mathcal{C}$, and N otherwise. Such facts are routinely confirmed in the absence of even a stray, evanescent thought about how M might or might not be embodied.

However, it's well-known that TMs (and other purely abstract computers) *can* be built. In fact, one such physical machine is processing the letters in the present sentence, as I (Selmer) type them. We may not know for sure that every abstract TM M^i from the countably infinite set of such devices can be physicalized to produce M_p^i, but certainly we *do* know that for every physical TM M_p^i able to accomplish some computation, there exists a corresponding purely mathematical TM that carries out the same computation (in the mathematical universe). This fact will prove convenient below.

Another well-known fact, one we also find rather helpful, is that there are simple physical processes *not reflective of the mathematical structure of TMs and the like*, which nonetheless solve

[1] See http://www.claymath.org/millennium. There are six other "millennium" problems; each of these is also associated with a $1M prize.

[2] As many readers know, the history of the problem is littered with failed attempts to provide non-constructive substantiation of the received view that **P**≠**NP**.

[3] His position is communicated in a stunningly prescient letter he wrote to von Neumann in 1950; this letter is reproduced, in English, in (Sipser 1992). Gödel, writing of course before the modern **P**=?**NP** framework, inquires as to von Neumann's thoughts about what is today known as the k-symbol provability problem. Let ϕ be a formula of $\mathcal{L}_I$ (a formula of first-order logic, or just FOL). We write $\vdash_k \phi$ provided there is a first-order proof of ϕ of $\leq k$ symbols. Gödel apparently believed that it might well be possible to answer questions of the form "$\vdash_k \phi$?" in linear or quadratic time. When the set here is made explicit and configured so as to allow for encoding on a Turing machine tape, it's patent that it's **NP**-complete. Gödel was quite at home with the idea that as logic and mathematics progress, machines would increasingly take over the "Yes-No" part of the enterprise. Any notion that Gödel would have embraced an argument by analogy from the undecidability of FOL to the perpetual intractability of the k-symbol provability problem is utterly misguided: He writes: "[I]t would obviously mean that in spite of the undecidability of the *Entscheidungsproblem*, the mental work of a mathematician concerning Yes-or-No questions could be completely replaced by a machine."

[4] An exactly parallel point obviously holds of all those incorporeal models known to be equivalent to TMs: register machines, the λ-calculus, abaci, etc. As more philosophically inclined readers will know, there isn't consensus as to what a physical instantiation of a mathematical machine *is*. However, given the orientation of the present paper, we can rise above such turbulence. The fact is, a cornerstone of computer science is that idealized machines can be physically realized: we teach computability theory because we presuppose this. For a sustained discussion of the relationship between abstract machines, and theorems regarding them, and their embodied counterparts, see (Bringsjord & Zenzen 1997).

some problems that are overwhelmingly difficult for TMs and their digital relatives. For example, the Steiner Tree problem (STP) is known to be **NP**-complete (see e.g. pp. 208–209 of Garey & Johnson 1979).[5] Nonetheless, a simple physical process (termed an **analog computation**[6]) can apparently solve it quickly. STP is the problem of connecting n points on a plane with a graph of minimal overall length, using junction points if necessary.[7] The physical process in question can be described in English as a straightforward algorithm: Make two parallel glass plates, and insert n pins between the plates to represent the points; dip the structure into a soap solution, and remove it; record the answer. The soap film will connect the n pins in the minimum Steiner-tree graph (Iwamura, Akazawa & Amemiya 1998).[8] Building the structure and the solution (and the container for the solution), activity usually termed *pre-processing* in analog computation, requires steps linear in the size of n, dipping and withdrawing make two steps, and recording — the so-called *post-processing* phase — is linear in n as well, so despite the fact that STP is **NP**-complete, the physical process just described — let's call it A^s — is apparently carried out well within $O(n^k)$, for some constant k.

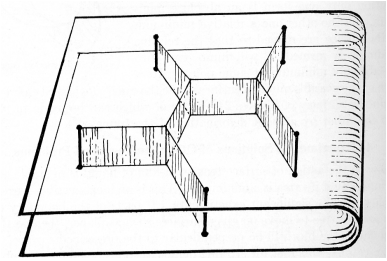

Figure 1: Solution to embodied Steiner Tree Problem with five points, found by soapfilm processing by Courant & Robbins (1941). The same results are obtained by Dewdney (1984).

At this point some readers may report that, at least for them, 'step' here has no precise meaning. They might say: "How is the process just described algorithmic? Why, for example, does it take one step to add a pin? Doesn't one have to measure the location (precisely!), drill a hole, fasten the pin, etc? After all, the construction is a more-or-less continuous process in space and time."

[5] STP is **NP**-hard when the metric is non-discretized, and **NP**-complete when the metric is discrete.

[6] Analog computers are nothing new, though they don't get much air time these days. An elegant example is Vannevar Bush's famous differential analyzer, which solves ordinary differential equations. A nice discussion of the analyzer (in connection not with Bush, but rather with Claude Shannon) can be found in (Earman 1986). A colorful discussion of the speedup that can be achieved through analog computation is provided by Dewdney (1984), who describes a series of analog computers, including something close to the soapfilm computer central to the present paper, and who also in some sense pointed in at least the general direction of our argument, by writing:

> Throughout the foregoing discussion [of analog computers for small inputs] I have dodged the important issue of the feasibility of constructing [such] gadgets. Although each of the gadgets can be built and persuaded to work, after a fashion, on small problems, it would be silly to suggest that one construct them with serious computations in mind. *Yet, considered in the context of an ideal world in which ideal materials are available, each gadget works, by definition, exactly as described.* (Dewdney 1984, p. 25; emphasis ours)

One of the oldest discussions of analog computers is presented by Courant & Robbins (1941), who tinkered with various wireframe-in-soapfilm analog computers, in connection with STP, and other problems.

[7] Just as we have the metaphor-clothed 'Traveling Salesmen Problem' (an **NP**-complete problem analyzed e.g. in Lewis & Papadimitriou 1981), STP can be conceived as the problem of building a road system (possibly with intersections outside the towns themselves) that connects n towns, where than system is of minimum length.

[8] As mentioned in note 6, additional descriptions of the analog soapfilm computation can be found, e.g., in (Courant & Robbins 1941, Dewdney 1984, Fischler & Firschein 1987).

There are two reasons why this sort of objection fails. First, Turing machines (and their equivalents) are subject to the same suspicions. After all, a Turing machine's read/write head must move from one square to the next, and each such move is a "step." Furthermore, rest assured that once a TM is embodied, it doesn't immediately pass into the non-algorithmic realm. On the contrary, embodied machines, though they run through "continuous processes in space and time," are behaving algorithmically. In addition, in Turing's (1936) original paper on the halting problem, he equates the behavior of the inanimate machine with the work of a 'computist' working with ordinary paper and pencil at the level used to couch the aforementioned soapfilm computation. The second reason the objection here is surmounted is given below, when the digital physics scheme is invoked — a scheme according to which every high-level process corresponds to a low-level one carried out by some Turing machine.

Returning to the soap process: Before starting to read this short paper, you were probably positive that **P≠NP**. If you've now heard about it for the first time, does the soap process change your mind? We didn't think so. But please reason further with us.

First, some simple notation. Let's refer to the physical version of the STP problem as $B(STP)$ (for *emBodied* STP), and the abstract version as STP. In addition, following usage above, we use M with or without superscripts to refer to Turing machines, and M_p to refer to physicalized TMs (with plain 'M' denoting abstract TMs). We refer to analog processes with variable A. Now here is a naïve proof, functioning as precursor to the more sophisticated successor given later, formalizable in sorted[9] first-order logic (FOL), that **P=NP** (where the predicate letter N is explained later):

The Preliminary, Naïve Proof

1	$\exists M$ (M solves STP in polynomial time) $\rightarrow$ **P=NP**	definition of **NP**-completeness
2	$\exists A$ (A solves $B(STP)$ in polynomial time $\wedge N(A)$)	derivable, e.g., by existential introduction from soapfilm process, i.e., A^s
3	$\exists A$ (A solves $B(STP)$ in polynomial time $\wedge N(A)$) $\rightarrow$ $\exists M_p(M_p$ solves $B(STP)$ in polynomial time)	digital physics; see below
4	$\exists M_p(M_p$ solves $B(STP)$ in polynomial time) $\rightarrow$ $\exists M(M$ solves STP in polynomial time)	unassailable; see justification 3rd ¶
5	**P=NP**	1–4 (full FOL derivation trivial)

We don't hold that this argument is sound as it stands (that's why the sophisticated successor is coming, after all), but there is no question that the reasoning here can be certified as formally valid (e.g., using an automated proof checker). The only question is whether the premises are true. If they are, the problem is at long last solved in the affirmative. Are the premises true?

Line 2, note, isn't a premise, but rather an intermediate conclusion; however, there are two routes to this conclusion. As noted in the justification column, one possible inference to line 2 is from A^s, where this constant is replaced by the variable A, and existential introduction is used.[10] The second route takes account of what we regard to be self-evident: Surely A^s is just the tip of the iceberg, with myriad analog processes out there in our physical universe waiting to be discovered and harnessed (though presumably most will remain undetected for eternity). This view can be derived from a sampling assumption, according to which a finding like A^s must be a random sampling from some small proper subset of the set of all candidate processes available in the cosmos. We don't pursue this derivation herein. Interested readers should consult a parallel form

[9] E.g., 1 is short for $\exists x(A(x) \wedge M$ solves $\ldots)$.

[10] We assume for certification a natural deduction calculus, with rules for introducing and eliminating truth-functional connectives and quantifiers; a nice system of this sort for FOL is $\mathcal{F}$, from Barwise & Etchemendy 1999. For natural deduction in modal logic, consult (Konyndyk 1986).

of argument explored in theoretical physics (see e.g. Bostrom 2002), and should instantiate that form in the present domain with additional samples (e.g., the various analog computers discussed in Dewdney 1984).

We anticipate that some will be uncomfortable with the view that there exists a process A that accommodates ever greater values for n. In light of this, we move now to the more sophisticated of our two proofs: First, note that full specification of our proof in FOL does include universal quantification over $\mathcal{N}$, and a corresponding index for STP and $B(STP)$, as for example in what line 2, unpacked, becomes:

$$\exists A \forall n (\ldots B(STP)_n \ldots) \ldots$$

Now, it's exceedingly hard to see how A^s cannot succeed on $n+1$ if the minimal graph has been found for n physicalized points. Whatever underlying principles of physics generate the graph in the case of n surely can be employed to generate it for $n+1$. Even if the physical laws governing *our* universe are such that there is some point $n+1$ at which A^s fails, surely it's logically physically *possible* that this failure *not* occur.[11] In short, it's mathematically possible that there is a world $w^\star$ at which it's physically possible that the underlying physics working so well on some initial interval (e.g., 1–6) in our world w^α, works for the *entire* countably infinite set.[12]

It's easy enough to make the reasoning here more precise. Begin by abbreviating 'A solves $B(STP)_n$,' where A is some analog process, by $S(A, B(STP)_n$. Next, relativize this locution to a world, so that, for example, we know (by virtue of soapfilm tinkering in our world)

$$S(A^s, w^\alpha, B(STP)_n), 1 \leq n \leq 6.$$

We now need to express the progression we have in mind more carefully. The idea, again, is that as the size of the input grows, idiosyncrasies in the collection of physical laws governing the world w in question may prevent the soapfilm process from yielding the answer — but we can always be sure that if things work for n, in-principle adjustment will allow them to work for $n+1$, and so by induction the progression will work for all $k \in \mathcal{N}$. In our gedanken-experiment we simply add to and modify the collection of physical laws so that the underlying physics that worked for $n-1$ works for n. We refer to this augmentation and adjustment by way of the $\oplus$ function, which yields a world preserving the underlying physics (least action, perhaps; see note 12) that worked previously (in the world to the left of $\oplus$), combined with what it takes to reach the next increment.

So, we have

[11] The double modal operator is key. While $\models_w \Diamond_p \phi$ means in the standard formal semantics that ϕ is true at some possible world $w' \in \mathcal{W}_p \subsetneq \mathcal{W}$ (where $\mathcal{W}$ is the set of all possible worlds logically/mathematically accessible from w, and $\mathcal{W}_p$ the proper subset of $\mathcal{W}$ preserving the physical laws of w), and $\models_w \Diamond \phi$ holds iff ϕ is true at some world in $\mathcal{W}$, $\models_w \Diamond\Diamond_p \phi$ means that $\Diamond_p \phi$ holds at some world $w' \in \mathcal{W}$.

[12] Michael Zenzen has pointed out to us that the so-called Principle of Least Action is a prime candidate for something that is at the core of what is carried over from world to world in order to reach to $w^\star$ (for discussion of these principles, see the remarkable Castigliano 1966). This principle would be what Jim Fahey has referred to as a 'proto'-law of nature. It's a profound waste of time to build gadgets that, in w^α, compute ever larger initial intervals, if the aim of such activity is to seek evidence that counts either for or against our argument: The conditions in such seat-of-the-pants experiments are not controlled (even Courant (1941), when running his soapfilm experiments, realized the need to idealize), and all our proof needs is truth at quite-removed $w^\star$, *not* our actual world.

$$S(A^s, w^\alpha, B(STP)_n), 1 \leq n \leq 6$$
$$S(A^s, (w^\alpha \oplus w^2), B(STP)_7)$$
$$S(A^s, ((w^\alpha \oplus w^2) \oplus w^3), B(STP)_8)$$
$$\vdots$$
$$S(A^s, (\cdots((w^\alpha \oplus w^2) \oplus w^3) \oplus \cdots \oplus w^m), B(STP)_{m+5})$$
$$\vdots$$

Now we set $w^\star$ to

$$\sum_{1}^{\omega} \cdots ((w^1 \oplus w^2) \oplus w^3) \cdots$$

and infer by the ω-rule that

$$\forall n S(A^s, w^\star, B(STP)_n).$$

The foregoing allows us to formulate a more formidable second proof that employs modal logic (Chellas 1980, Hughes & Cresswell 1968, Konyndyk 1986). If we let $\Diamond_p$ refer to physical possibility in a manner that parallels the straight $\Diamond$ of logical possibility from modal logic,[13] then the modal version of line 3 is

$$\Diamond\Diamond_p \exists A \forall n (A \text{ solves } B(STP)_n \text{ in polynomial time } \wedge N(A)) \rightarrow$$

$$\Diamond\Diamond_p \exists M_p \forall n (M_p \text{ solves } B(STP)_n \text{ in polynomial time})$$

and this technique can be easily propagated through the original proof to produce the more circumspect one. In this modal proof, line 4 becomes the key principle that if it's logically physically possible that a physical TM solve $B(STP)_n$ in polynomial time, then there exists (in the mathematical universe) a TM that solves STP_n in polynomial time. This principle would appear to be invulnerable. Summing up, we have:

The Modalized Proof

$1'$	$\exists M \forall n (M$ solves STP_n in polynomial time$) \rightarrow$ **P**=**NP**	definition of **NP**-completeness
$2'$	$\Diamond\Diamond_p \exists A \forall n$ (A solves $B(STP)_n$ in polynomial time $\wedge N(A)$)	derivable by induction, the ω-rule, existential intro on A^s
$3'$	$\Diamond\Diamond_p \exists A \forall n$ (A solves $B(STP)_n$ in polynomial time $\wedge N(A)) \rightarrow$ $\Diamond\Diamond_p \exists M_p \forall n (M_p$ solves $B(STP)_n$ in polynomial time)	*possible* digital physics; see below
$4'$	$\Diamond\Diamond_p \exists M_p \forall n (M_p$ solves $B(STP)_n$ in polynomial time $\rightarrow$ $\exists M \forall n (M$ solves STP_n in polynomial time))	unassailable; see justification 3rd ¶
$5'$	**P**=**NP**	$1'$–$4'$ (full FOL derivation trivial)

Please note that the modal version of our argument provides complete immunity from an objection that the physical universe is finite, and that therefore no analog process can scale up through all natural numbers as inputs for the minimal graph to be generated. The dominant view among theoretical physicists appears to be that the theory of inflation (Vilenkin 1983, Guth 2000) holds (which renders it likely that the universe is infinite). But we need not take a stand on the issue. All we need is what follows immediately from the fact that the theory of inflation, whether or not true, is certainly coherent: namely, that it's logically physically *possible* that the universe is infinite.

[13] We assume a normal S5 version of the $\Diamond$ operator.

By "digital physics" in the justification of premise 3/3′, we have in mind the position that the physical universe is fundamentally a vast physical computer — or, if you like, a computer composed of computers, which are in turn composed of computers, and so on. This view has been recently affirmed by Wolfram (2002), but Fredkin (1990) advanced the view long ago (and continues to energetically defend it now), and Feynman (1982) seems to have embraced the view as well. Even Einstein can be read as having affirmed the digital physics position. Whether or not the actual world is fundamentally governed by digital physics, there can be no denying that there is a possible world that is — and this weak fact is all that our proof presupposes. Though premise 3/3′ refers to physicalized Turing machines, most digital models in physics are based on cellular automata, but this is of no matter: it's well-known that every cellular automaton can be recast as a TM.[14]

Our argument shows that if **P≠NP**, digital physics is incoherent. Since it must be true that all physical phenomena can in principle be modeled in information-processing terms of *some* kind, **P≠NP** thus immediately implies, courtesy of our arguments, that hypercomputational processes exist in some physical universes.[15] If you believe, as many do, that hypercomputational processes are always merely mathematical, and never physically real, you can't be rational and at the same time refuse to accept our case for **P=NP**.

Perhaps you do indeed refuse to accept the coherence of the digital physics view, and have no qualms about in-principle physical hypercomputation. It was with skeptics like you in mind that the predicate $N(\ )$ (for "normal") was included in our proofs. While many are perhaps right to point out, *contra* Wolfram and company, that some physical phenomena (e.g., those associated with quantum mechanics) are so bizarre and complicated that they resist formalization in TM-level computational models, the fact of the matter is that the analog process we exploit is a painfully simple macroscopic phenomenon — as we say, a "normal" physical process. There are two important features implied by normality in the present case.

First, the soapfilm process (and for that matter any of the many known analog processes that yield speedup, e.g., the "string" computer: Dewdney 1984, Dewdney 1985) shouldn't be treated any different than ordinary physical processes interpreted to fall on the scalable digital side. We can use an abacus functioning as a Register machine, or a model railroad system functioning as a Turing machine, to see that we have on hand a process that, at least in principle, solves a problem. When the abacus is correctly deployed to add $2 + 2$ and hence leave 4 beads in the designated location, we know that, assuming the physics to be preserved as we scale up, $n + n$ presents no trouble.

The second important feature purchased by normality is that it precludes any cheating with respect to hardware. By definition, a guess for an **NP** problem is checkable in polynomial time. Likewise, by definition, if exponential hardware is allowed, the check can be carried out in polynomial time. In a *non*-normal situation, the underlying physical stuff can be imagined to grow even at a geometric rate, and so a computer can in polynomial time "solve" problems that grow at a

[14]The transformation preserves polynomial-time processing, as cognoscenti know. For others, a sketch: Use an n-dimension TM in which n is high enough to sufficiently represent the CA which is the universe. Let each cell of the TM's tapes represent a cell of the CA. The alphabet of the TM will contain some representation of all states for the CA's cells. The computation performed by the CA is finite, so the TM's states are as well. It is known that the transformation from multidimensional Turing machines to standard Turing machines is a polynomial transformation. If the computation performed in each cell of the CA is in class **P**, the equivalent TM will be in class **P**.

[15]Physical phenomena that can be rigorously modeled only via information processing above the **Turing Limit** (Siegelmann 1999, Bringsjord & Zenzen 2003) would be phenomena calling for hypercomputational machines. Just as the class of mathematical devices equivalent to TMs is infinite, so also there are an infinite number of hypercomputational machines. Examples include **analog chaotic neural nets** (Siegelmann & Sontag 1994) and **infinite time Turing machines** (Hamkins & Lewis 2000). Other examples include **analog "knob" TMs** (Bringsjord 2001) and **accelerated TMs** (Copeland 1998).

geometric rate.[16] The soapfilm process leverages physical material in the standard way. We don't have a supply of soapy water assumed to grown exponentially as the number of pins increases. In fact, the great irony facing anyone who claims that we are sneaking in exponentially increasing hardware is that the empirical and arithmetical facts (e.g., see Dewdney 1984) indicate that analog computation derives its power from the ability to handle larger and larger input while its core process (happening after what we've called pre-processing) hums along at linear time on hardware whose size grows in linear fashion.

The bottom line regarding normality is that the burden of proof is surely on those who would maintain that the formal machinery of digital physics is in principle insufficient to model something as straightforward as submerging nails in, and retrieving them from, a bucket of soapy water.

Postscript

A reader asks:

> "Some rather smart people continue to feverishly search for object-language level confirmation of the received view that P≠NP. Do you seriously maintain that this work is otiose? What if you woke up tomorrow to find that someone had established, at the object-language level, what people generally expect: namely, P≠NP?"

If we received this information tomorrow, and the proof checked out, we would be faced with an antinomy, since our argument certainly *appears* to be sound — and the conclusion of a sound argument comes no less highly recommended than the conclusion of a proof. Admittedly, this would be an antinomy Bringsjord has imagined since the argument given above occurred to him, for the simple reason that, intuitively, he is *himself* inclined to believe that **P≠NP** — while on the other hand, again, there stands the argument in question, and there really is no denying its power. It's also possible that humans, not just us, but *all* humans, will forever fail to wake to such a proof, for the reason that the question is independent of classical mathematics, as identified with some axiomatic set theory cast in first-order logic. In this case, the future wouldn't bring the result our interlocutor asks us to imagine, but rather only, for eternity, meta-arguments on the issue. If so, here is ours, given at the dawn of the coming dialectic.

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[16] The cardinality of sets produced by successive application of the power set operator $\mathcal{P}$ grows geometrically, but we can imagine 2^n processors that share memory in which the input $I = \{i_1, i_2, \ldots, i_n\}$ is placed, each of which is connected to a printer. Starting at i_1 and working up in parallel, each processor does or doesn't issue a print command, depending upon whether the i_k matches its unique identifier. In linear time $\mathcal{P}(I)$ is printed.

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