Theorem:
General intelligence entails creativity, assuming ...*

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Contents

*I'm deeply grateful for the opportunity to speak in Montpelier at C3GI 2012, a lecture that marked my pursuit, seemingly a pipe dream then, for a rigorous rationale in favor of what was then confessedly only an inchoate suspicion: that general intelligence (when both high and "alien-fair") entails creativity. I have benefitted from conversations with Naveen Sundar G. and Matthias Scheutz, and am grateful to them as well. Finally, the support of AFOSR has been crucial. Needless to say, =any errors appearing herein are mine alone, unrelated to any of the kind minds who have interacted with me on matters treated herein.

1 Apparent Evidence Against the Claim

There appears to be considerable evidence against the claim that general intelligence implies creativity. For example, with this claim unpacked as the proposition that any general-intelligent agent must be creative, the field of AI declares the claim to be false. To see this, we need but note that the dominant and encyclopedic AI textbook, (Russell & Norvig 2009), defines an intelligent agent as one that computes a mapping from its percepts to its actions. The range of potential mappings explored in this volume is non-trivial, ranging from simple arithmetic functions to functions that leverage declarative knowledge, and beyond. But never is creativity discussed in connection with any of these functions; indeed creativity is nowhere discussed in the book, period; nor for that matter is any synonym (e.g., 'innovative') discussed. In short, as far as this highly influential and comprehensive volume is concerned, general-intelligent agents needn't be creative.

Of course, *AIMA*, as it's known, is a textbook, at the end of the day; a masterful one, yes, but certainly a textbook. It's on the bookshelf of nearly every single AI researcher and engineer on our planet, but the tome doesn't purport to provide a novel account of general machine intelligence. Yet it seems to me that we observe the same lack-of-entailment result if we examine "research-grade" proposals for what abstract machine intelligence is. One example is Hutter (2005) theory of "universal artificial intelligence." Whatever virtues this theory may have (and I do think it has some significant ones), an explanation of creativity isn't one of them. Hutter's formal foundations are avowedly and indeed proudly in sequential decision theory and algorithmic information theory; but such things, if the scientific literature on creativity is any guide, would be top candidates for being *in tension* with creativity. Part of the reason for this is presumably that if we know anything about creativity in the human case, and from that know something about the abstract concept of creativity that can cover information-processing machines and extraterrestrial lifeforms, we know that creativity leverages declarative knowledge to produce new concepts, from which new declarative knowledge is generated. The paradigm of this "creativity engine" at work is the evolution of mathematics and mathematical knowledge. If Leibniz hadn't used what he did know to create the concept of an infinitesimal, what we know in knowing analysis via knowing the theorems that constitute it, might never have arrived. So in Hutter's work we have a proposal for what the nature of intelligence is, in the abstract — but nothing in that proposal yields that general intelligence entails creativity.

The somewhat odd thing, though, is that Hutter (2005) does *mention* creativity, and indeed does so in a context that seems quite relevant to the present essay. For we read:

> The science of [AI] might be defined as the construction of intelligent systems and their analysis. A natural definition of a *system* is anything that has an input and an output stream. Intelligence is more complicated. It can have many faces like **creativity**, solving problems, pattern recognition, classification, learning, induction, deduction, building analogies, optimization, surviving in an environment, language processing, knowledge, and many more. A formal definition incorporating every aspect of intelligence, however, seems difficult. Further, intelligence is graded ... So, the best we can expect to find is a partial or total order relation on the set of systems, which orders them w.r.t. their degree of intelligence (like intelligence tests do for human systems, but for a limited class of problems). Having this order we are, of course, interested in large elements, i.e., highly intelligent systems. (Hutter 2005, 2–3; bold text from me)

The contrast between this and AIMA is quite interesting. AIMA defines an agent, as we've noted, as an input-output device, with inputs as percepts and outputs as actions. So what Hutter says about systems fits the AIMA framework well. But then the list of "faces" that he gives, and casts aside as infeasible targets for targeted formalization, include many things that AIMA in fact provides computational definitions of (save, as we've noted, for creativity!). I see no reason to despair of formalizing all of these parts of human cognition, and for the life of me don't understand why Hutter rules such a project out as too difficult. The problem that *I* see, from the standpoint of truly general intelligence, abstracted away from us and our machines to cognizers in general, is that many of these parts of human cognition aren't necessarily part of highly intelligent cognizers in the abstract case. I take up this problem below (§2.1), and suggest a solution.

It's also interesting to note that Hutter is to this point roughly in line with what I shall propose, which is a hierarchy of intelligence (and one inspired by my psychometric tendencies, which renders Hutter's comment about human intelligence tests welcome) — but for reasons that remain utterly mysterious to me, he takes maximization of some utility function to be the essence of intelligence, to which all the "faces" he lists are supposed to be reducible. He writes: "Most, if not all, known facets of intelligence can be formulated as goal driven or, more precisely, as maximizing some utility function." (Hutter 2005, 3). But no proof or argument is offered in support of this credo. This is all the more disturbing in light of the fact that on some accounts of creativity, for instance on some interpretations of what

Boden (1991) calls *P-creativity*, to be creative is to somehow produce something which cannot be understood from, let alone derived from, antecedents (e.g. see Bringsjord, Ferrucci & Bello 2001). In the case of Leibniz, Zeno's paradoxes of motion stood iron-strong for century upon century, and then suddenly new infinitary concepts arrive on the scene, and soon thereafter ordinary physical motion makes perfect mathematical sense. At any rate, whether or not Hutter is right, the fact remains that while his *Universal Artificial Intelligence* is a certified research-grade proposal for what general intelligence, in man or machine, is, the book's Index contains no entry for creativity; and therefore at the very least we have no reason to think, on the basis of Hutter's book that what I'm declaring to be a theorem is one.

Let's try a third tack. In keeping with so-called *Psychometric AI* (PAI, rhymes with 'π') (Bringsjord & Schimanski 2003, Bringsjord 2011, Bringsjord & Licato 2012), according to which AI consists in the engineering of artificial agents capable of high performance on well-defined tests of various vaunted mental powers in the human sphere, we can quickly see that, once again, general intelligence doesn't seem to entail creativity: Let a be an agent able to perfectly answer every question on every established, psychometrically validated test of general (human) intelligence.[1] And now pull off the shelf every single established test of creativity used by psychometricians and psychologists.[2] Next, does our assumption of a's prowess enable us to deduce that a will score at a high level on the selected test of creativity? No. Indeed, the negative here is so obvious, and so firm, that I will not trouble the reader with any details, and will instead sum up the psychometric chasm between tests of general human intelligence and tests of human creativity by giving this telling, representative fact: It doesn't follow from the proposition that some agent is able to achieve perfection at digit recall[3] that that agent can quickly invent new things to do with tin cans.[4]

2 The Setup

Despite the foregoing, in point of fact it *is* possible to show that high general intelligence, whether of the human, alien, or machine variety, *does* entail cre-

[1]Two ideal picks would be Raven's Progressive Matrices (Raven 1962), and the WAIS (discussed in connection with AI in Bringsjord & Schimanski 2003).

[2]One good choice would be the Torrance Tests of Creative Thinking (TTCT) (Torrance 1990), targeted in the AI work reported in (Bringsjord & Ferrucci 2000).

[3]Digit recall is a sub-test on the WAIS (see note 1. On this sub-test, the test-taker attempts to repeat back a string of digits given to him by the tester.

[4]A typical question on TTCT. See note 2.

ativity — as long as certain assumptions are made. I find these assumptions to be eminently reasonable, but lay no claim that they in fact are: Readers are invited to judge for themselves. As this "theorem" is a conditional, a natural way to construct the proof of it is to assume, with respect to arbitrary instances of all the categories over which quantifiers in the theorem range, the relevant antecedent, and then derive the consequent. This is the route I take. But please note that as the present essay is intended to be digestible by a general educated audience, I provide herein only an informal proof-*sketch*, not a full-blown proof. And I seek to make the presentation here largely self-contained: I assume only that my readers have had but a bit of elementary mathematical logic and recursion theory. If you had this, but have forgotten it, your memory will soon be refreshed.

We turn now to setting out our preliminaries. This will be a review for many readers; for others it will serve to secure the self-contained nature of the essay.

2.1 A Focus on Arithmetic

I begin by setting some context: Since we are operating under the framework of PAI, the objective is a proof that if some agent a is general-intelligent under an "alien-fair" test $\mathcal{T}_{gi}$ of general intelligence, then a, under some "alien-fair" test $\mathcal{T}_c$ of creativity, is creative.

What is $\mathcal{T}_{gi}$, the alien-fair test of general intelligence (= gi)? It should be obvious that if our test of general intelligence is to be not only culture-fair relative to the cultures in place on Earth, but also a fair test of gi for any spot in the universe, and indeed for any place in any universe, we can't base $\mathcal{T}_{gi}$ on anything that is clearly just a part of our local environment as human beings. The solution is to restrict the test of gi to something that every single class or race of general-intelligent agents must to an appreciable degree master: arithmetic.

Notice that I don't say 'mathematics.' Rather, I specifically refer to arithmetic. This is because clearly parts of the vast edifice of human-discovered mathematics might not be tackled by general-intelligent aliens. For example, extraterrestrials on Alpha Centauri (assuming they are there for the sake of exposition), however brilliant they may be, might never take up geometry. But there would seem to be absolutely no way these aliens can avoid seeking and securing arithmetical competence. A genuinely general-intelligent alien agent, as well as an information-processing agent that we or such an alien brings into existence, couldn't dodge arithmetic, and the search for substantial knowledge of it. But we must be a bit more systematic about what arithmetic is, and how much of it must be mastered if an agent is to earn the right to be classified as gi. The next step is to quickly review some standard formal machinery from mathematical logic and computability.[5]

2.2 Basic Machinery

Let $\mathbf{A}$ be some axiomatic theory of arithmetic based on some corresponding formal language $\mathscr{L}_{\mathbf{A}}$. Let $\mathscr{I}_{\mathcal{A}}$ be the natural and received model-theoretic interpretation of ordinary arithmetic with which you and I are intimately familiar. (Warning: I use $\mathbf{A}$ as a variable for a given axiom system, instances to be visited below; but I use $\mathcal{A}$ to refer to arithmetic, period.) $\mathbf{A}$ is simply a set of formulae from $\mathscr{L}_{\mathbf{A}}$: viz., a certain set of **axioms** of arithmetic. Now let $\alpha_{\mathbf{A}} \in \mathscr{L}_{\mathbf{A}}$ be some arbitrary formula about $\mathbf{A}$ arithmetic. (When the context is clear, we shall sometimes drop the subscript $_{\mathbf{A}}$ and refer to a given arithmetical formula as simply α.) To say that α is true on some interpretation $\mathscr{I}$, we write the customary:

$$\mathscr{I} \models \alpha$$

Where $\Phi_{\mathbf{A}}$ (here too we sometimes omit the subscript $_{\mathbf{A}}$) is a set of formulae based on $\mathscr{L}_{\mathbf{A}}$, α is a **consequence** of Φ iff

$$\text{For every } \mathscr{I}, \text{ if all of } \Phi \text{ are true on } \mathscr{I}, \text{ then } \mathscr{I} \models \alpha$$

$\mathbf{A}^{\models}$ denotes the set of all formulae that are consequences of $\mathbf{A}$. We assume a standard finitary proof-theory τ based in first-order logic (e.g., resolution-, natural deduction-, or equational-based), and write the usual

$$\Phi \vdash_{\tau} \alpha$$

to indicate that α is provable from Φ in this theory. $\mathbf{A}^{\vdash_{\tau}}$ denotes the set of all theorems that can be proved from $\mathbf{A}$ in the proof. Finally, we set

$$\text{TRUE}_{A/X} = \{\alpha \in \mathscr{L}_{\mathbf{A}} : \mathscr{I}_{\mathcal{A}} \models \alpha\}$$

[5] My notation and focus is devised for the purposes at hand, but in general nice coverage is provided in the venerable (Ebbinghaus, Flum & Thomas 1994), which I have long used in classroom teaching of intermediate mathematical logic. But there is an especially good background provided in (Smith 2013), which has the added benefit of a learned discussion of potential ways of distinguishing between an understanding of basic arithmetic, versus understanding more. Unlike Smith (2013), I have high-ish standards: I interpret basic arithmetic to include truths of arithmetic beyond ordinary, mechanical proof in first-order logic.

where X is a placeholder index enabling reference to modifications in the underlying language $\mathscr{L}_{\mathbf{A}}$ and axiom system $\mathbf{A}$. For instance, writing $\text{TRUE}_{A/I}$ will denote the set of all first-order arithmetical formulae true on the standard first-order interpretation $\mathscr{I}_{\mathcal{A}}$ of arithmetic.

2.3 Context for the Theorem

Our context is set out diagrammatically in Figure 1. The reader will be able to make sense of much of this figure given his/her assimilation of the review provided in §2.2. Here's the remaining explanation that is needed:

First, a circle C inside another circle C' indicates a proper subset relation; that is, $C \subset C'$. The contents of each circle is just a set of formulae.

Four axiom systems of arithmetic appear in the diagram. At the inner core, in the smallest circle, the system **EA**, "elementary arithmetic," appears. More precisely, every theorem (consequence) of the axioms **EA** is what composes the innermost circle. Smith (2013) refers to this axiom system as "Baby Arithmetic," and I follow suit, and so deploy '**BA**,' as can be seen the diagram. No one could take **BA** seriously as an axiom system that captures what even moderately intelligent pre-teen humans know about arithmetic. For example, while any true instance of an equation of the form $n+m=k$ is deducible from **BA**, and likewise any true instance of an equation of the form $n \times m = k$ as well, **BA** is severely limited, since for instance it doesn't even allow formulae and deduction with the quantifiers $\exists$ and $\forall$. There is thus little point here in saying anything further about **BA**, since even moderately intelligent schoolchildren know truths of arithmetic that make use of variables and (implicit) quantifiers (e.g., $\forall x(x \times 1 = x)$, which such children would recognize as $x \times 1 = x$). In other words, an alien-fair $\mathcal{T}_{gi}$ would have to include questions like the following, which aren't in the innermost circle.

Q $\forall x(x \times 1 = x)$?

In Figure 1, I indicate that an agent in the innermost circle understands everything within this circle. I do this by depicting the "face" of the agent inside this circle. But notice that there is an arrow flowing from this picture of the agent that leaves the innermost circle and travels to the immediate superset, that is, to the next circle. And notice that this arrow has a check on it. What this says is that any agent of moderate intelligence who has reached pre-college development will not only understand **BA**, but also, if they have an understanding of basic arithmetic, **Q**, or — as it is sometimes known — **Robinson Arithmetic**.

The axioms of **Q** do allow quantification, and can be associated with the standard proof theory τ that I invoked above.[6] While the above question Q is settled by **Q**, this axiom system isn't powerful enough to support the test $\mathcal{T}_{gi}$. This is clear from the fact that for example the following question would be easy for a young student, but features a theorem that isn't in $\mathbf{Q}^{\vdash_\tau}$.

Q' $\forall x(0 + x = x)$?

Obviously, then, we can't identify alien-fair gi with the second circle in Figure 1. In light of this, note that I give the agent in the diagram a free pass to the next circle, which is labeled with 'TRUE$_{A/\mathbf{Q}}$, the set of all formulae that are true on the standard interpretation of arithmetic, relativized to **Q**. Obviously $\forall x(0 + x = x)$ is in TRUE$_{A/\mathbf{Q}}$.

We come now to the third axiom system: $\mathbf{PA}_I$. This is standard Peano Arithmetic, which many readers will at least have heard something about. It includes the first six of the seven axioms composing **Q** (see note 6), plus one additional axiom schema:

Induction Schema Every sentence that is the universal closure of an instance of this schema:

$$[\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))] \rightarrow \forall x\phi(x)$$

$\mathbf{PA}_I$ is what I shall take as the springboard from which to launch to a completion of the definition of alien-fair gi.[7] We can jump from the circle

[6] **Q** is composed of seven axioms (where s is the successor or "increment-by-one" function):

Axiom 1 $\forall x(0 \neq s(x))$

Axiom 2 $\forall x \forall y(s(x) = s(y) \rightarrow x = y)$

Axiom 3 $\forall x(x \neq 0 \rightarrow \exists y(x = s(y)))$

Axiom 4 $\forall x(+(x, 0) = x)$

Axiom 5 $\forall x \forall y(+(x, s(y)) = s(+(x, y)))$

Axiom 6 $\forall x(\times(x, 0) = 0)$

Axiom 7 $\forall x \forall y(\times(x, s(y)) = +(\times(x, y), x))$

[7] Alert readers will note that in jumping from this board I pass straight through $\mathbf{ACA}_0$ without comment, and they will have already have noticed the dotted circle I drew for this axiom system. What gives? Ultimately, $\mathbf{ACA}_0$ supports a class of first-order theorems that doesn't exceed those provable from $\mathbf{PA}_I$; hence the dotted circle rather than a solid one. As to what $\mathbf{ACA}_0$ is, we shall have to rest content with the highly informal piece of information that it's a restricted form of second-order arithmetic. A philosophically rich presentation of $\mathbf{ACA}_0$ is provided in (Smith 2013).

containing all the theorems provable from $\mathbf{PA}_I$ (which is the same set as all the consequences of $\mathbf{PA}_I$) to a larger class: namely, all the truths of standard first-order arithmetic. (Circles beyond this one involve second-order logic (as indicated by the subscript $_{II}$), and are left for future expansions of my case for the gi-implies-creativity theorem.) But notice that here the arrow designed to reflect the "travel" of our agent is not labeled with a check, but rather with a question-mark. This is so because making this jump requires some impressive intelligence. What do you reach if you make this jump, specifically? I give four examples in the diagram, each one marked with a $\star$. And what are the examples? I will cite only two here, in the interests of space. The first is $\mathcal{G}$, and is a label for the formulae that Gödel pointed to via his first incompleteness theorem. Each of these formulae is of course such that neither it nor it's negation can be proved from $\mathbf{PA}_I$ — but each such formula is true on $\mathscr{I}_{\mathcal{A}}$. The second $\star$ proposition I call out here is a particular number-theoretic fact: Goodstein's Theorem (Goodstein 1944), and is indicated by 'GT.' While GT and instances of $\mathcal{G}$ are all true on the standard interpretation of arithmetic, they are beyond the theorems of $\mathbf{PA}_I$, a nice result first proved by Kirby & Paris (1982).[8]

2.4 Key Definitions

I define a truly gi agent, whether human, alien, or machine, to be one that understands not only basic arithmetic (i.e., that understands $\mathbf{PA}_I^{\vdash_\tau}$ and below, to include $\therefore$ $\mathbf{Q}^{\vdash_\tau}$ and $\mathbf{EA}^{\vdash}/\mathbf{BA}^{\vdash_\tau}$), but also at least one $\star$ truth.

The test $\mathcal{T}_{gi}$ is composed of truths of arithmetic; that is, of members of the set $\text{TRUE}_{A/I}$, with an accompanying request for a supporting proof. In the case of $\mathbf{PA}_I$, there can be a supporting formal proof in a standard, finitary, mechanizable proof theory (our τ). This will not be possible for members of the set

$$\text{TRUE}_{A/I} - \mathbf{PA}_I^{\vdash_\tau}$$

I further define creativity as passing beyond such proofs in basic arithmetic in order to reach at least one $\star$ truth. Again, Goodstein's Theorem is currently an ideal example.

[8] GT is simply the fact that a particular sequence of natural numbers, the **Goodstein sequence**, starting with any natural number n, eventually terminates at zero. But many folks who first understand the sequence are utterly convinced that it's both astonishingly fast-growing and never terminates, and simply returns larger and larger numbers as the sequence progresses, forever. See (Potter 2004) for a nice version of the proof, which makes use of infinitary concepts and techniques, and turns these intuitions upside down to yield a result that a truly general-intelligent agent can appreciate.

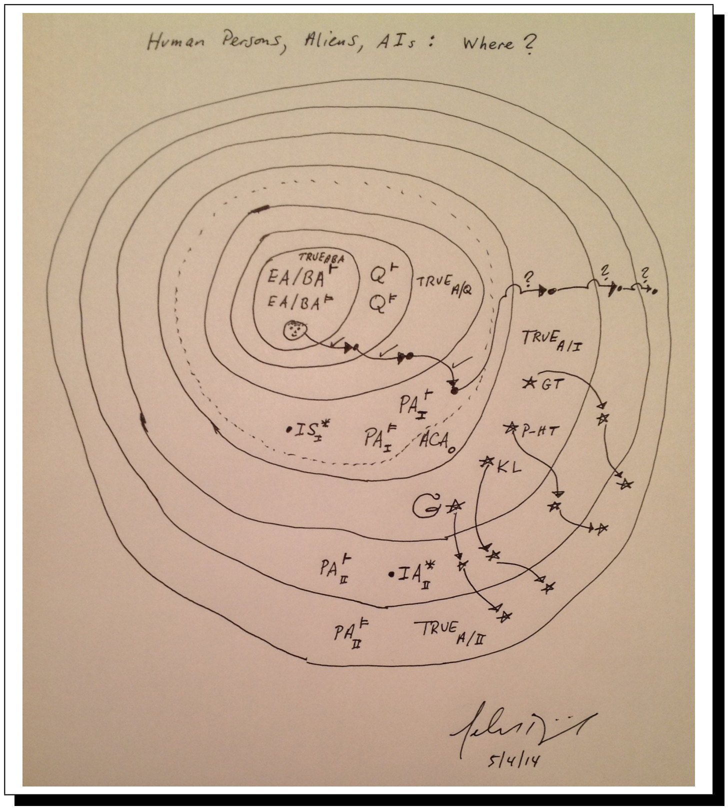

Figure 1: The Context for the Theorem

3 The Proof-Sketch Itself

Theorem: Suppose that an agent a is alien-fair gi. Then a is creative.

Proof-Sketch: Trivial, given our setup. Assume the hypothesis of the theorem. By definition, a, since it's gi includes command of all of basic arithmetic, knows at least one $\star \in \mathrm{TRUE}_{A/I}$ on the strength of a proof π discovered and confirmed by a. But since π as a matter of mathematical fact exceeds the mechanical type of proof that characterizes our τ, a has left behind mere mechanical, first-order techniques, and is by definition creative. **QED**

4 Objections

Some objections can be anticipated; I discuss two.

4.1 Begging the Question?

Objection: "As you yourself note, given your setup, the theorem is easily established. So you have simply begged the question. Why would anyone accept your setup in the first place?"

My reply: Well, every theorem presupposes background machinery, and some of it will be objectionable to some; the present situation is no exception. I cheerfully admit that anyone unwilling to accept that alien-fair gi must include significant command over basic arithmetic at the level of the truths of arithmetic will not be persuaded. But I maintain that at the very least it's undeniable that it's not unreasonable to construe alien-fair gi in such a manner that understanding of one or more $\star$ is included. After all, Earth-bound empirical evidence is on my side, given the remarkable creativity it has taken to reach some $\star$ truths. I also maintain that it's not unreasonable to identify creativity with a process of coming to know some $\star$, since invariably this hard-won knowledge comes via reasoning that is beyond the rigid, mechanistic construction of standard formal proofs in first-order logic.

4.2 A Non-Creative Route to a $\star$?

Objection: "But here's a *non*-creative way to reach the performance you say is creative, which serves as a counter-example to your so-called theorem: We know that the set of all formulae reachable from the grammar and alphabet of $\mathscr{L}_{\mathbf{A}}$ by standard recursive rules for well-formedness is countably infinite. Even the extensions of the language to make room for the moves to second-order logic of course stay within the bound of countably infinite. Hence there is a machine $\mathcal{M}$ which prints out (in accordance with some lexicographic ordering) the first such formula, then the second, and so on. In addition, $\mathcal{M}$ is assumed to be equipped with a random "formula picker" P such that, given a formula in the relevant class, it returns either TRUE or FALSE randomly. Clearly, if $\mathcal{M}$-plus-P is lucky, it will declare all the $\star$'s you're talking about to be true — and yet clearly this 'agent' is operating in purely mechanistic, naïve fashion, indeed more so than the searching for proofs in the proof theory τ."

My reply: Multiple problems are fatal to this objection; I mention two here. First, if this objection worked, then basic incompleteness results like $\mathcal{G}$ would in some sense be surmounted as well, by stunningly "dumb" means. But no one thinks there's a shortcut here to establishing formulae that are beyond $\mathbf{PA}^{\vdash_\tau}$. Second, my tests of both gi and creativity (in the realm of arithmetic) are such that to pass requires *understanding*, and the behavioral correlate to understanding, taken to confirm its presence, is *justification*. In other words, and this repeats what has been said above, to pass $\mathcal{T}_{gi}$, an agent must prove that their answers to basic arithmetic are correct; and to pass $\mathcal{T}_c$ must prove at least one $\star$ truth about arithmetic. The dim contraption $\mathcal{M}$-plus-P does nothing of the sort.

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