Gödel’s First Incompleteness Theorem
(excerpted from Gödel’s Great Theorems)

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Intro to Logic
May 2 2016
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Thursday:

Can a machine match Gödel?

Grade roundup (not today; let us sort out the recent mis-sendings first), & contest “outflow.”
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Goal: Put you in position to prove Gödel’s first incompleteness theorem!
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We have the background (if).
Remember …
“The (Economical) Liar” …
which will be our guide!
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L: This sentence is false.
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Suppose that \( T(L) \); then \( \neg T(L) \).
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Suppose that $T(L)$; then $\neg T(L)$.

Suppose that $\neg T(L)$ then $T(L)$. 
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L: This sentence is false.

Suppose that $T(L)$; then $\neg T(L)$.

Suppose that $\neg T(L)$ then $T(L)$.

Hence: $T(L)$ iff (i.e., if & only if) $\neg T(L)$. 
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Suppose that $\neg T(L)$ then $T(L)$.

Hence: $T(L)$ iff (i.e., if & only if) $\neg T(L)$.

Contradiction!
Next, recall:

**PA (Peano Arithmetic):**

\[
\begin{align*}
A1 \quad & \forall x (0 \neq s(x)) \\
A2 \quad & \forall x \forall y (s(x) = s(y) \rightarrow x = y) \\
A3 \quad & \forall x (x \neq 0 \rightarrow \exists y (x = s(y))) \\
A4 \quad & \forall x (x + 0 = x) \\
A5 \quad & \forall x \forall y (x + s(y) = s(x + y)) \\
A6 \quad & \forall x (x \times 0 = 0) \\
A7 \quad & \forall x \forall y (x \times s(y) = (x \times y) + x)
\end{align*}
\]

And, every sentence that is the universal closure of an instance of

\[
([\phi(0) \land \forall x (\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall x \phi(x))
\]

where \(\phi(x)\) is open wff with variable \(x\), and perhaps others, free.
Arithmetic is Part of All Things Sci/Eng/Tech!

and courtesy of Gödel: We can’t even prove all truths of arithmetic!

Each circle is a larger part of the formal sciences.
Definition of Richard’s $N$:

“The real number whose whole part is zero, and whose $n$-th decimal is $p$ plus one if the $n$-th decimal of the real number defined by the $n$-th member of $E$ is $p$ and $p$ is neither eight nor nine, and is simply one if this $n$-th decimal is eight or nine.”

Proof: $N$ is defined by a finite string taken from the English alphabet, so $N$ is in the sequence $E$. But on the other hand, by definition of $N$, for every $m$, $N$ differs from the $m$-th element of $E$ in at least one decimal place; so $N$ is not any element of $E$. Contradiction! QED
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And remember the “dictionary” sequence in Richard’s Paradox.

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Gödel Numbering, the Easy Way

Easy peasy: Just realize that every entry in a dictionary is named by a number $n$, and by the same basic lexicographic ordering, every computer program, formula, etc. is named by a number $m$ in a lexicographic ordering going from 1, to 2, to …
Easy peasy: Just realize that every entry in a dictionary is named by a number \( n \), and by the same basic lexicographic ordering, every computer program, formula, etc. is named by a number \( m \) in a lexicographic ordering going from 1, to 2, to …

So, \textit{gimcrack} is named by some positive integer \( k \). Hence, I can just refer to this word as “\( k \)".
Gödel’s First Incompleteness Theorem

Suppose that elementary arithmetic (i.e., PA) is consistent (no contradiction can be derived in it) and program-decidable (there’s a program \( P \) that, given as input an arbitrary formula \( p \), can decide whether or not \( p \) is in PA). Then there is sentence \( g^* \) in the language of elementary arithmetic which is such that:

\( g^* \) can’t be proved from PA (i.e., not PA \( |- g^* \))!

And, not-\( g^* \) can’t be proved from PA either (i.e., not PA \( |- \text{not-}g^* \))!
Proof Kernel for Theorem Gl

Part I: Recipe $R$

Let $q(x)$ be an arbitrary formula of arithmetic with one open variable $x$. (E.g., $x + 3 = 5$. And here $q(2)$ would be $2 + 3 = 5$.)

Gödel invented a recipe $R$ that, given any $q(x)$ as an ingredient template that you are free to choose, produces a self-referential formula $g$ such that:

$$\text{PA } \vdash g \iff q(\text{"g"})$$

(i.e., a formula $g$ that says: “I have property $q$!”)
First, for $q(x)$ we choose a formula $q^*$ that holds of any “s” if and only if $s$ can be proved from PA; i.e.,

$$\text{PA} \vdash q^*("s") \iff \text{PA} \vdash s \quad (1)$$

Next, we follow Gödel’s Recipe $R$ to build a $g^*$ such that:

$$\text{PA} \vdash g^* \iff \neg q^*("g^*"), \quad (2)$$
Proof Kernel for Theorem GI

Part 2: Follow Recipe $R$, Guided by The Liar

First, for $q(x)$ we choose a formula $q^*$ that holds of any “$s$” if and only if $s$ can be proved from $\text{PA}$; i.e.,

$$\text{PA} \vdash q^*("s") \iff \text{PA} \vdash s \quad (1)$$

Next, we follow Gödel’s Recipe $R$ to build a $g^*$ such that:

$$\text{PA} \vdash g^* \iff \neg q^*("g^*)" \quad (2)$$

$g^*$ thus says: “I’m not true!” And so, the key question (assignment!): $\text{PA} \vdash g^*?!?$
Austin Hardyesque Indirect Proof

GI: \( g^* \) isn’t provable from PA; nor is the negation of \( g^* \! \\

Proof: Let’s follow The Liar: Suppose that \( g^* \) is provable from PA; i.e., suppose \( \text{PA} \vdash g^* \). Then by (1), with \( g^* \) substituted for \( s \), we have:

\[
\text{PA} \vdash q^*(“g^*”) \iff \text{PA} \vdash g^* \quad (1’)
\]

From our supposition and working right to left by *modus ponens* on (1’) we deduce:

\[
\text{PA} \vdash q^*(“g^*”) \quad (3.1)
\]

But from our supposition and the earlier (see previous slide) (2), we can deduce by *modus ponens* that from PA the opposite can be proved! I.e., we have:

\[
\text{PA} \vdash \text{not-}q^*(“g^*”) \quad (3.2)
\]

But (3.1) and (3.2) together means that PA is inconsistent, since it generates a contradiction. Hence by indirect proof \( g^* \) is *not* provable from PA.
Austin Hardy-esque Indirect Proof

GI: $g^*$ isn’t provable from PA, nor is the negation of $g^*$.

Proof: Let’s follow The Liar: Suppose that $g^*$ is provable from PA; i.e., suppose $\text{PA} |- g^*$. Then by (1), with $g^*$ substituted for $s$, we have:

$$\text{PA} |- q^*\left(\text{“}g^*\text{”}\right) \iff \text{PA} |- g^* \quad (1')$$

From our supposition and working right to left by modus ponens on (1’) we deduce:

$$\text{PA} |- q^*\left(\text{“}g^*\text{”}\right) \quad (3.1)$$

But from our supposition and the earlier (see previous slide) (2), we can deduce by modus ponens that from PA the opposite can be proved! I.e., we have:

$$\text{PA} |- \neg q^*\left(\text{“}g^*\text{”}\right) \quad (3.2)$$

But (3.1) and (3.2) together means that $\text{PA}$ is inconsistent, since it generates a contradiction. Hence by indirect proof $g^*$ is not provable from PA.

What about the second option? Can you follow The Liar to show that supposing that the negation of $g^*$ (i.e., not-$g^*$) is provable from PA also leads to a contradiction, and hence can’t be? (Good Test 3 question?)
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