Rebuilding the Foundations of Math via (the “Theory”) ZFC; ZFC to Axiomatized Arithmetic (the “Theory” PA)

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Intro to Logic
3/22/2018
Dear colleague,

For a year and a half I have been acquainted with your Grundgesetze der Arithmetik, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [Moment] in logic and when you place a high value upon an ideography [Begriffsschrift] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your Begriffsschrift), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [p. 23 above]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let \( w \) be the predicate: to be a predicate that cannot be predicated of itself. Can \( w \) be predicated of itself? From each answer its opposite follows. Therefore we must conclude that \( w \) is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.\(^1\) I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore\(^1\) I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your Grundgesetze; I hope that this will still be done.

Very respectfully yours,

BERTRAND RUSSELL

The above contradiction, when expressed in Peano's ideography, reads as follows:

\[ w = \text{cls} \cap x \exists (x \sim e x).\overset{\sim}{:} w \in w \neq w. \sim e w. \]
Dear colleague,

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\[
\begin{align*}
   w &= \text{cls} \cap x \exists (x \sim e \ x). \therefore w \in w \therefore w \sim e w.
\end{align*}
\]
Russell’s Theorem
Russell’s Theorem

\[ \vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y) \]
Russell’s Theorem

\[ \vdash \neg \exists x \forall y (y \in x \leftrightarrow y \not\in y) \]

(Poor Frege!)
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\[ \vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y) \]

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http://plato.stanford.edu/entries/russell-paradox/#HOTP
The Foundation Crumbles

The Rest of Math, Engineering, etc.

Foundation

Axiom V etc.
The Foundation Crumbles

The Rest of Math, Engineering, etc.

Foundation
The Foundation Crumbles

The Rest of Math, Engineering, etc.

Foundation
The Foundation Crumbles

The Rest of Math, Engineering, etc.

Foundation
Last time: Russell’s Paradox
Last time: Russell’s Paradox

Now: Richard’s Paradox …
a
b
.
.
.
.
aa
ab
.
.
.
.
.

Doesn’t define a real number.
Doesn’t define a real number.
Definition of Richard's $N$:

Doesn't define a real number.

$E$
Definition of Richard’s $N$:

“The real number whose whole part is zero, and whose $n$-th decimal is $p$ plus one if the $n$-th decimal of the real number defined by the $n$-th member of $E$ is $p$ and $p$ is neither eight nor nine, and is simply one if this $n$-th decimal is eight or nine.”
Definition of Richard’s $N$:

“The real number whose whole part is zero, and whose $n$-th decimal is $p$ plus one if the $n$-th decimal of the real number defined by the $n$-th member of $E$ is $p$ and $p$ is neither eight nor nine, and is simply one if this $n$-th decimal is eight or nine.”

Proof: $N$ is defined by a finite string taken from the English alphabet, so $N$ is in the sequence $E$. But on the other hand, by definition of $N$, for every $m$, $N$ differs from the $m$-th element of $E$ in at least one decimal place; so $N$ is not any element of $E$. Contradiction! QED
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Suppose $N$ is $u_m$. 
Definition of Richard’s $N$:
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Suppose $N$ is $u_m$.

Suppose $u_m$ is $0.xxxx...xxx...
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Suppose $u_m$ is $0.xxxx…xxx…$

Suppose $u_m$ is $0.xxxx…8xx…
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Suppose $N$ is $u_m$.

Suppose $u_m$ is $0.xxxx…xxx…$

Suppose $u_m$ is $0.xxxx…8xx…$

Then $N$ is $0.xxxx…1xx…$
Definition of Richard’s $N$:

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Suppose $N$ is $u_m$.

Suppose $u_m$ is $0.\ldots0xx\ldots$

Suppose $u_m$ is $0.\ldots8xx\ldots$

Then $N$ is $0.\ldots1xx\ldots$

Hence $N$ can’t be $u_m$!
Definition of Richard’s $N$:

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Suppose $N$ is $u_m$.

Suppose $u_m$ is $0.xxxx\ldots.xxx\ldots$

Suppose $u_m$ is $0.xxxx\ldots8xx\ldots$

Suppose $u_m$ is $0.xxxx\ldots5xx\ldots$

Then $N$ is $0.xxxx\ldots1xx\ldots$

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Suppose $N$ is $u_m$.

Suppose $u_m$ is $0.xxxx\ldots xxx\ldots$

Suppose $u_m$ is $0.xxxx\ldots 8xx\ldots$

Suppose $u_m$ is $0.xxxx\ldots 5xx\ldots$

Then $N$ is $0.xxxx\ldots 1xx\ldots$

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Suppose $N$ is $u_m$.

Suppose $u_m$ is $0.\ldots xxx\ldots$

Then $N$ is $0.\ldots 1xx\ldots$

Hence $N$ can’t be $u_m$!
Definition of Richard’s $N$:
“The real number whose whole part is zero, and whose $n$-th decimal is \textcolor{red}{p plus one} if the $n$-th decimal of the real number defined by the $n$-th member of $E$ is $p$ and $p$ is neither eight nor nine, and is simply one if this $n$-th decimal is eight or nine.”

Suppose $N$ is $u_m$.

Suppose $u_m$ is $0.xxxx…xxx…$

Suppose $u_m$ is $0.xxxx…8xx…$
Then $N$ is $0.xxxx…1xx…$
Hence $N$ can’t be $u_m$!

Suppose $u_m$ is $0.xxxx…5xx…$
Then $N$ is $0.xxxx…6xx…$
Hence $N$ can’t be $u_m$!
The Foundation Rebuilt

The Rest of Math, Engineering, etc.

New Foundation
The Foundation Rebuilt

The Rest of Math, Engineering, etc.

New Foundation

ZFC
The Foundation Rebuilt

The Rest of Math, Engineering, etc.

New Foundation

ZFC
The Foundation Rebuilt

The Rest of Math, Engineering, etc.

Arithmetic

New Foundation

ZFC
The Foundation Rebuilt

The Rest of Math, Engineering, etc.

New Foundation

Arithmetic

ZFC

So what are the axioms in ZFC?
Russell’s Paradox … to ZFC

\[ \vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y) \]

(Russell’s Theorem; poor Frege!)

http://plato.stanford.edu/entries/russell-paradox/#HOTP

(reveals even this version of Slate as real-life tool)

Supplant Cantor’s/Frege’s Axiom V with the Axiom
Schema of Separation (& put on our thinking caps …) and you try to show Theorem 1 from Suppes:

\[ \vdash \forall x (x \notin \emptyset) \]
Russell’s Paradox … to ZFC

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\[ \vdash \forall x (x \notin \emptyset) \]

You try a second “Suppesian” theorem in ZFC:
Russell’s Paradox … to ZFC

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You try a second “Suppesian” theorem in ZFC:

\[ \vdash \forall x [(\forall z (z \notin x)) \rightarrow x = \emptyset] \]
Russell’s Paradox … to ZFC

\[ \vdash \neg \exists x \forall y (y \in x \iff y \notin y) \]

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You try a second “Suppesian” theorem in ZFC:

\[ \vdash \forall x [(\forall z (z \notin x)) \rightarrow x = \emptyset] \]

Now let’s add the Definition of Subset to ZFC:
Russell’s Paradox … to ZFC

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You try a second “Suppesian” theorem in ZFC:

\[ \vdash \forall x [(\forall z (z \notin x)) \rightarrow x = \emptyset] \]

Now let’s add the Definition of Subset to ZFC:

\[ \forall x \forall y [x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y)] \]
Russell’s Paradox … to ZFC

\[ \vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y) \]

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Now let’s add the Definition of Subset to ZFC:

\[ \forall x \forall y [x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y)] \]

With this definition, can you prove (Theorem 3) that every set is a subset of itself?
Formal Arithmetic ...
\( \mathbf{Q} \) (= Robinson Arithmetic)

A1 \( \forall x (0 \neq s(x)) \)
A2 \( \forall x \forall y (s(x) = s(y) \rightarrow x = y) \)
A3 \( \forall x (x \neq 0 \rightarrow \exists y (x = s(y)) \)
A4 \( \forall x (x + 0 = x) \)
A5 \( \forall x \forall y (x + s(y) = s(x + y)) \)
A6 \( \forall x (x \times 0 = 0) \)
A7 \( \forall x \forall y (x \times s(y) = (x \times y) + x) \)
Q (= Robinson Arithmetic)

A1 \( \forall x (0 \neq s(x)) \)
A2 \( \forall x \forall y (s(x) = s(y) \rightarrow x = y) \)
A3 \( \forall x (x \neq 0 \rightarrow \exists y (x = s(y)) \)
A4 \( \forall x (x + 0 = x) \)
A5 \( \forall x \forall y (x + s(y) = s(x + y)) \)
A6 \( \forall x (x \times 0 = 0) \)
A7 \( \forall x \forall y (x \times s(y) = (x \times y) + x) \)
PA (Peano Arithmetic)

A1  ∀x (0 ≠ s(x))
A2  ∀x ∀y (s(x) = s(y) → x = y)
A3  ∀x (x ≠ 0 → ∃y (x = s(y)))
A4  ∀x (x + 0 = x)
A5  ∀x ∀y (x + s(y) = s(x + y))
A6  ∀x (x × 0 = 0)
A7  ∀x ∀y (x × s(y) = (x × y) + x)

And, every sentence that is the universal closure of an instance of

( [φ(0) ∧ ∀x (φ(x) → φ(s(x)))] → ∀x φ(x) )

where φ(x) is open wff with variable x, and perhaps others, free.
PA (Peano Arithmetic)

A1  \( \forall x (0 \neq s(x)) \)
A2  \( \forall x \forall y (s(x) = s(y) \rightarrow x = y) \)
A3  \( \forall x (x \neq 0 \rightarrow \exists y (x = s(y)) \)
A4  \( \forall x (x + 0 = x) \)
A5  \( \forall x \forall y (x + s(y) = s(x + y)) \)
A6  \( \forall x (x \times 0 = 0) \)
A7  \( \forall x \forall y (x \times s(y) = (x \times y) + x) \)

And, every sentence that is the universal closure of an instance of

\[
([\phi(0) \land \forall x (\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall x \phi(x))
\]

where \( \phi(x) \) is open wff with variable \( x \), and perhaps others, free.
Selmer, what’s this open WFF concept?
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We’ve already seen it in our coverage of ZFC.
Selmer, what’s this open wff concept?

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\[ \exists y [s(s(0)) \times y = s(s(s(s(0))))] \]
Selmer, what’s this open wff concept?

We’ve already seen it in our coverage of ZFC.

$$\exists y [s(s(0)) \times y = s(s(s(s(0))))]$$

This says what?
Selmer, what’s this open wff concept?

We’ve already seen it in our coverage of ZFC.

\[ \exists y [ s(s(0)) \times y = s(s(s(s(0)))) ] \]

This says what?

That 2 multiplied by some number yields 4.
Selmer, what’s this open wff concept?

We’ve already seen it in our coverage of ZFC.

$$\exists y [s(s(0)) \times y = s(s(s(s(0))))]$$

This says what?

That 2 multiplied by some number yields 4.

But this is very specific: the successor of the successor of zero is specifically 2.
Selmer, what’s this open wff concept?

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This says what?

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But this is very specific: the successor of the successor of the successor of zero is specifically 2.

Here then is the general case with an “open” wff:
Selmer, what’s this open wff concept?

We’ve already seen it in our coverage of ZFC.

\[ \exists y \left[ s(s(0)) \times y = s(s(s(s(0)))) \right] \]

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Here then is the general case with an “open” wff:

\[ \exists y[s(s(0)) \times y = x] \]

This open wff \( \phi(x) \) expresses the arithmetic property ‘even.’
Again: Do we just manufacture mathematics?
Again: Do we just manufacture mathematics?
Again: Do we just manufacture mathematics?

No, we tap into deep, underlying reality — and aliens do/would too …
Again: Do we just manufacture mathematics?

No, we tap into deep, underlying reality — and aliens do/would too …
Objection from Astrologic: Aliens Will be on the Same “Race Track”!
Slutten