The Liar; Russell’s Paradox

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Intro to Logic
3/19/2018
Types of Paradoxes

- Deductive Paradoxes
- Inductive Paradoxes — coming (e.g. The Lottery Paradox & The St Petersburg Paradox)
Types of Paradoxes

First:

- Deductive Paradoxes

- Inductive Paradoxes — coming (e.g. The Lottery Paradox & The St Petersburg Paradox)
The Liar (Paradox) …
The (Economical) Liar
The (Economical) Liar

L: This sentence is false.
The (Economical) Liar

**L:** This sentence is false.

If $T(L)$ then $\neg T(L)$
The (Economical) Liar

L: This sentence is false.

If \( T(L) \) then \( \neg T(L) \)

If \( \neg T(L) \) then \( T(L) \)
The (Economical) Liar

L: This sentence is false.

If T(L) then ¬T(L)

If ¬T(L) then T(L)

T(L) iff (i.e., if & only if) ¬T(L)
The (Economical) Liar

L: This sentence is false.

If $T(L)$ then $\neg T(L)$

If $\neg T(L)$ then $T(L)$

$T(L)$ iff (i.e., if & only if) $\neg T(L)$

Contradiction!
The (Verbose) Liar — With a Twist
The (Verbose) Liar — With a Twist

Theorem: 2 + 2 = 5.
The (Verbose) Liar — With a Twist

Theorem: 2+2 = 5.

Proof: Set:
The (Verbose) Liar — With a Twist

Theorem: $2+2 = 5$.

Proof: Set:

$L$: This sentence is false.
The (Verbose) Liar — With a Twist

Theorem: 2 + 2 = 5.

Proof: Set:

L: This sentence is false.

L is either true or false. Suppose that it’s true. Then since what it says is that it’s false, it is false; i.e., L is false, on this supposition. So we’ve proved that if L is true, L is false. Now suppose instead that L is false. Then since it says that it’s false, it’s true; i.e., L is true, on our current supposition. We have thus proved that if L is false, L is true. Combining the conditionals we’ve proved yields this: L is true if and only if L is false, which is a contradiction. (P if and only if ¬P is logically equivalent to P and ¬P.) By inference schema explosion, it follows that 2 + 2 = 5. QED
Outlawing Self-Referential Sentences Isn’t the Answer!
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- For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
Outlawing Self-Referential Sentences Isn’t the Answer!

- For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
- This sentence is a sentence.
Outlawing Self-Referential Sentences Isn’t the Answer!

- For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.

- This sentence is a sentence.

- This sentence contains the letter ‘r’.
Outlawing Self-Referential Sentences Isn’t the Answer!

- For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
  - This sentence is a sentence.
  - This sentence contains the letter ‘r’.
  - This sentence has more than three letters in it.
Outlawing Self-Referential Sentences Isn’t the Answer!

• For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
  • This sentence is a sentence.
  • This sentence contains the letter ‘r’.
  • This sentence has more than three letters in it.
  • This sentence ends with a period, starts with a capital ’T’, and has more than two words.
Outlawing Self-Referential Sentences Isn’t the Answer!

• For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.

• This sentence is a sentence.

• This sentence contains the letter ‘r’.

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• This sentence ends with a period, starts with a capital ’T’, and has more than two words.

• ...
Outlawing Self-Referential Sentences Isn’t the Answer!
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Box 1
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1

Box 2
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1

The sentence in Box 2 is true.

Box 2

The sentence in Box 1 is false.
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1

The sentence in Box 2 is true.

Box 2

Neither sentence is self-referential.

The sentence in Box 1 is false.
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1

The sentence in Box 2 is true.

Neither sentence is self-referential.

Box 2

The sentence in Box 1 is false.
Outlawing Self-Referential Sentences Isn’t the Answer!

Suppose that the sentence in Box 1 is true. Then the sentence in Box 2 is true (because the sentence in Box 1 says that that sentence is true). But then the sentence in Box 1 is false (because the sentence in Box 2 says that that sentence is false). So, if the sentence in Box 1 is true, it’s false. On the other hand, by parallel deduction, if the sentence in Box 1 is false, the sentence in Box 1 is true. (Make sure you work out and verify the reasoning that establishes the previous sentence.) We thus have again a contradiction: The sentence in Box 1 is true if and only if it’s not true.
Further Reading …
Russell’s Paradox …
Dear colleague,

For a year and a half I have been acquainted with your Grundgesetze der Arithmetik, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [Moment] in logic and when you place a high value upon an ideography [Begriffsschrift] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§ 9 of your Begriffsschrift), I have been led on my own to views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17 [p. 23 above]) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let \( w \) be the predicate: to be a predicate that cannot be predicated of itself. Can \( w \) be predicated of itself? From each answer its opposite follows. Therefore we must conclude that \( w \) is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.

I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly.\(^1\) I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.

The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore\(\text{I have permitted myself to express my deep respect to you. It is very }
\text{regrettable that you have not come to publish the second volume of your Grund-}
\text{gesetze}; I hope that this will still be done.}

Very respectfully yours,

Bertrand Russell

The above contradiction, when expressed in Peano's ideography, reads as follows:

\[
w = \text{cls} \cap x \exists (x \sim e x).	herefore w e w \therefore w \sim e w.
\]
Dear colleague,

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$$w = \text{cls} \cap x \exists(x \sim e x). \vdash w e w : = w \sim e w.$$
The Foundation Crumbles

The Rest of Math, Engineering, etc.

Foundation

Axiom V etc.
The Foundation Crumbles

The Rest of Math, Engineering, etc.

Foundation
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Foundation
There was once a small town in Norway in which there was a barber who shaved all and only the men residing in the town who didn’t shave themselves.
There was once a small town in Norway in which there was a barber who shaved all and only the men residing in the town who didn’t shave themselves.

There was once a small town in Norway in which there resided a barber who shaved all and only the men residing in the town who didn’t shave themselves.
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There was once a small town in Norway in which there resided a barber who shaved all and only the men residing in the town who didn’t shave themselves.

There was once a small town in Norway in which there resided a male barber who shaved all and only the men residing in the town who didn’t shave themselves.
Such a situation is impossible!
Such a situation is impossible!

**Proof:** Let’s assume for the sake of argument that such a situation can be. Without loss of generality, let the town be Lyngdal and the male Lyngdalian barber be Olaf. Either Olaf shaves himself or he doesn’t. But either case leads straight to a contradiction. Therefore the situation is in fact impossible. Here we go …

Suppose Olaf shaves himself. Then it follows that he doesn’t shave himself. Suppose on the other hand that Olaf doesn’t shave himself. Then it follows that he does shave himself. Hence, Olaf shaves himself if and only if he doesn’t shave himself, which is a contradiction. **QED**
Russell’s Theorem
Russell’s Theorem

\[ \vdash \neg \exists x \forall y (y \in x \leftrightarrow y \notin y) \]
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(Poor Frege!)
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http://plato.stanford.edu/entries/russell-paradox/#HOTP
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