Number Theory & Astrologic; Second-Order Logic and the k-order Ladder; Second-Order Axiomatized Arithmetic; Gödel’s Speedup Theorem; Time Travel

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Intro to Logic
4/2/2018
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Intro to Logic
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\( Q \)  
\( (= \text{Robinson Arithmetic}) \)

\begin{align*}
A1 \quad & \forall x (0 \neq s(x)) \\
A2 \quad & \forall x \forall y (s(x) = s(y) \rightarrow x = y) \\
A3 \quad & \forall x (x \neq 0 \rightarrow \exists y (x = s(y))) \\
A4 \quad & \forall x (x + 0 = x) \\
A5 \quad & \forall x \forall y (x + s(y) = s(x + y)) \\
A6 \quad & \forall x (x \times 0 = 0) \\
A7 \quad & \forall x \forall y (x \times s(y) = (x \times y) + x) 
\end{align*}
Q\(^\text{= Robinson Arithmetic}\)

A1 \(\forall x (0 \neq s(x))\)

A2 \(\forall x \forall y (s(x) = s(y) \rightarrow x = y)\)

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A6 \(\forall x (x \times 0 = 0)\)

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PA (Peano Arithmetic)

A1 \( \forall x (0 \neq s(x)) \)
A2 \( \forall x \forall y (s(x) = s(y) \rightarrow x = y) \)
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A6 \( \forall x (x \times 0 = 0) \)
A7 \( \forall x \forall y (x \times s(y) = (x \times y) + x) \)

And, every sentence that is the universal closure of an instance of

\[
([\phi(0) \land \forall x (\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall x \phi(x))
\]

where \( \phi(x) \) is open wff with variable \( x \), and perhaps others, free.
PA (Peano Arithmetic)

A1 \( \forall x (0 \neq s(x)) \)
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And, every sentence that is the universal closure of an instance of

\[ ([\phi(0) \wedge \forall x (\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall x \phi(x)) \]

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PA (Peano Arithmetic)

A1  \( \forall x (0 \neq s(x)) \)
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Astrologic

(Aliens & Angels on the Same “Race Track”)
Astrologic
(Aliens & Angels on the Same “Race Track”)

An intellectual steeple for human persons!
Astrologic
(Aliens & Angels on the Same “Race Track”)

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An intellectual steeple for human persons!
FOL
✓ FOL

Epistemic + FOL

$B_d B_v B_d V_v$
\checkmark \text{FOL}

\text{Epistemic + FOL} \quad (\text{for coverage of “killer” robots})

B_dB_vB_dV_u
√ FOL

√ Epistemic + FOL  \( B_d B_v B_d V_u \)  

(for coverage of “killer” robots)
✓ FOL

✓ Epistemic + FOL \[ B_d B_v B_d V v \]

TOL

\[ \exists X [X(j) \land \neg X(m) \land S(X)] \] (for coverage of "killer" robots)
✓ FOL
✓ Epistemic + FOL

\[B_d B_v B_d V v\]

TOL

\[\exists X [X(j) \land \neg X(m) \land S(X)]\]  

(for coverage of “killer” robots)
✓ FOL

✓ Epistemic + FOL

$B_d B_v B_d V v$

TOL

$\exists X [X(j) \land \neg X(m) \land S(X)]$

(for coverage of “killer” robots)
Double-Minded Man
Double-Minded Man
Double-Minded Man

68-year old Harriet Smith sits with two wrinkled hands firmly on the wheel of her rust-eaten Subaru wagon, staring straight ahead through the top level of bifocals as she waits serenely at a red light.

Harriet is alone in the car. To her right is another vehicle, also waiting, in this case to make a right turn; it's a sleek, low-slung, black Camaro.

We are inside the cabin with Harriet. The Subaru's sound system softly plays choral music. Harriet's lips move slightly as she internally sings along, mouthing a slow aria. Her head weaves slightly side to side, in the rhythm with the music.

Things are calm as can be here inside the car with Harriet. There are a pair of well-worn Bibles on the empty passenger seat beside her, one with a gold-lettered 'Harriet' on its leather front cover, the other with a matching 'Joseph' on its front cover.

Harriet's eyes swivel up to the light, still red. We wait with her.

Suddenly there is a piercing SCREECH outside. Harriet jerks her head to the right and we follow her line of sight.

A sleek motorcycle has swerved out of its lane and is now streaking straight for the right side of the Camaro beside Harriet's car.

The bike slams with CLANG into the side of the Camaro. Its rider is flung up and forward into the air, twirling passed Harriet's windshield.

We now watch from Harriet's pov, in slow motion. The black leather-clad motorcyclist sits by Harriet's windshield, stillborn. We see a man's face clearly. His elephanthide skin tells us that he is well beyond middle age. Yet thick, black curls of youthful hair emerge from under his helmet. The rider has only half a face of a black, bushy, swept-out, waxed mustache. His eyes are weary and grey, and appear to lock with Harriet's for an instant.

We return to normal speed. The body is now lying on the incoming lane to the left of Harriet's Subaru, perfectly still on the blacktop, the head twisted into an impossible angle. Blood seeps from a nostril. Beside the lifeless head, a BMW medallion lies on the pavement, glinting in the sunlight.
Double-Minded Man
Double-Minded Man

by

S Bringsjord & A Bringsjord

DRAFT #5
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Double-Minded Man
1. TWIRL - DAY

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Double-Minded Man

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Double-Minded Man

\[ \exists X [ X(\text{joseph}) \land \neg X(m(\text{harriet, joseph})) \land \text{Sleazy}(X)] \]
Double-Minded Man

\[ \exists X [ X(joseph) \land \neg X(m(harriet, joseph)) \land Sleazy(X)] \]
Climbing the $k$-order Ladder
Climbing the $k$-order Ladder

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

$Llama(a) \land Llama(b) \land Likes(a, b) \land Llama(fatherOf(a))$

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

\[ \text{ZOL} \quad \text{Llama}(a) \land \text{Llama}(b) \land \text{Likes}(a, b) \land \text{Llama} (\text{fatherOf}(a)) \]

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

$\text{ZOL} \quad Llama(a) \land Llama(b) \land Likes(a, b) \land Llama(fatherOf(a))$

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

$\exists x [\text{Llama}(x) \land \text{Llama}(b) \land \text{Likes}(x,b) \land \text{Llama}(\text{fatherOf}(x))]$

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

FOL  \[ \exists x[Llama(x) \land Llama(b) \land Likes(x, b) \land Llama(fatherOf(x))] \]

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

ZOL  \[ Llama(a) \land Llama(b) \land Likes(a, b) \land Llama(fatherOf(a)) \]

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

Things $x$ and $y$, along with the father of $x$, share a certain property (and $x$ likes $y$).

**FOL** $\exists x[\text{Llama}(x) \land \text{Llama}(b) \land \text{Likes}(x, b) \land \text{Llama}(\text{fatherOf}(x))]$

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

**ZOL** $\text{Llama}(a) \land \text{Llama}(b) \land \text{Likes}(a, b) \land \text{Llama}(\text{fatherOf}(a))$

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

\[
\exists x \exists y \exists R[R(x) \land R(y) \land \text{Likes}(x, y) \land R(\text{fatherOf}(x))]
\]

Things $x$ and $y$, along with the father of $x$, share a certain property (and $x$ likes $y$).

**FOL**
\[
\exists x \left[ \text{Llama}(x) \land \text{Llama}(b) \land \text{Likes}(x, b) \land \text{Llama}(\text{fatherOf}(x)) \right]
\]

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

**ZOL**
\[
\text{Llama}(a) \land \text{Llama}(b) \land \text{Likes}(a, b) \land \text{Llama}(\text{fatherOf}(a))
\]

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

**SOL**

$$\exists x \exists y \exists R[R(x) \land R(y) \land \text{Likes}(x, y) \land R(\text{fatherOf}(x))]$$

Things $x$ and $y$, along with the father of $x$, share a certain property (and $x$ likes $y$).

**FOL**

$$\exists x [\text{Llama}(x) \land \text{Llama}(b) \land \text{Likes}(x, b) \land \text{Llama}(\text{fatherOf}(x))]$$

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

**ZOL**

$$\text{Llama}(a) \land \text{Llama}(b) \land \text{Likes}(a, b) \land \text{Llama}(\text{fatherOf}(a))$$

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

Things $x$ and $y$, along with the father of $x$, share a certain property; and, $x$ $R^2$s $y$, where $R^2$ is a positive property.

**SOL**  
$\exists x \exists y \exists R[R(x) \land R(y) \land Likes(x, y) \land R(fatherOf(x))]$

**FOL**  
$\exists x[Llama(x) \land Llama(b) \land Likes(x, b) \land Llama(fatherOf(x))]$

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

**ZOL**  
$Llama(a) \land Llama(b) \land Likes(a, b) \land Llama(fatherOf(a))$

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the $k$-order Ladder

$\exists x, y \exists R, R^2[R(x) \land R(y) \land R^2(x, y) \land \text{Positive}(R^2) \land R(\text{fatherOf}(x))]$

Things $x$ and $y$, along with the father of $x$, share a certain property; and, $x$ $R^2$'s $y$, where $R^2$ is a positive property.

SOL $\exists x \exists y \exists R[R(x) \land R(y) \land \text{Likes}(x, y) \land R(\text{fatherOf}(x))]$

Things $x$ and $y$, along with the father of $x$, share a certain property (and $x$ likes $y$).

FOL $\exists x[Llama(x) \land Llama(b) \land \text{Likes}(x, b) \land Llama(\text{fatherOf}(x))]$

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

ZOL $Llama(a) \land Llama(b) \land \text{Likes}(a, b) \land Llama(\text{fatherOf}(a))$

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
Climbing the \( k \)-order Ladder

**TOL** \( \exists x, y \exists R, R^2[R(x) \land R(y) \land R^2(x, y) \land \text{Positive}(R^2) \land R(\text{fatherOf}(x))] \)

Things \( x \) and \( y \), along with the father of \( x \), share a certain property; and, \( x \) \( R^2 \)'s \( y \), where \( R^2 \) is a positive property.

**SOL** \( \exists x \exists y \exists R[R(x) \land R(y) \land \text{Likes}(x, y) \land R(\text{fatherOf}(x))] \)

Things \( x \) and \( y \), along with the father of \( x \), share a certain property (and \( x \) likes \( y \)).

**FOL** \( \exists x[Llama(x) \land Llama(b) \land \text{Likes}(x, b) \land Llama(\text{fatherOf}(x))] \)

There's some thing which is a llama and likes \( b \) (which is also a llama), and whose father is a llama too.

**ZOL** \( Llama(a) \land Llama(b) \land \text{Likes}(a, b) \land Llama(\text{fatherOf}(a)) \)

\( a \) is a llama, as is \( b \), \( a \) likes \( b \), and the father of \( a \) is a llama as well.
Climbing the $k$-order Ladder:

TOL $\exists x, y \exists R, R^2[R(x) \land R(y) \land R^2(x, y) \land \text{Positive}(R^2) \land R(\text{fatherOf}(x))]$

Things $x$ and $y$, along with the father of $x$, share a certain property; and, $x$ $R^2$'s $y$, where $R^2$ is a positive property.

SOL $\exists x \exists y \exists R[R(x) \land R(y) \land \text{Likes}(x, y) \land R(\text{fatherOf}(x))]$

Things $x$ and $y$, along with the father of $x$, share a certain property (and $x$ likes $y$).

FOL $\exists x[Llama(x) \land Llama(b) \land \text{Likes}(x, b) \land Llama(\text{fatherOf}(x))]$

There's some thing which is a llama and likes $b$ (which is also a llama), and whose father is a llama too.

ZOL $Llama(a) \land Llama(b) \land \text{Likes}(a, b) \land Llama(\text{fatherOf}(a))$

$a$ is a llama, as is $b$, $a$ likes $b$, and the father of $a$ is a llama as well.
\[ \text{PA} = \mathbb{Z}_0 \]

\begin{align*}
\text{A1} & \quad \forall x (0 \neq s(x)) \\
\text{A2} & \quad \forall x \forall y (s(x) = s(y) \to x = y) \\
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\text{A4} & \quad \forall x (x + 0 = x) \\
\text{A5} & \quad \forall x \forall y (x + s(y) = s(x + y)) \\
\text{A6} & \quad \forall x (x \times 0 = 0) \\
\text{A7} & \quad \forall x \forall y (x \times s(y) = (x \times y) + x)
\end{align*}

And, every sentence that is the universal closure of an instance of

\[ ([\phi(0) \land \forall x (\phi(x) \to \phi(s(x)))] \to \forall x \phi(x)) \]

where \( \phi(x) \) is open wff with variable \( x \), and perhaps others, free.
PA = Z₀

A1  ∀x(0 ≠ s(x))
A2  ∀x∀y(s(x) = s(y) → x = y)
A3  ∀x(x ≠ 0 → ∃y(x = s(y)))
A4  ∀x(x + 0 = x)
A5  ∀x∀y(x + s(y) = s(x + y))
A6  ∀x(x × 0 = 0)
A7  ∀x∀y(x × s(y) = (x × y) + x)

And, every sentence that is the universal closure of an instance of

(ϕ(0) ∧ ∀x(ϕ(x) → ϕ(s(x)))) → ∀xϕ(x)

where ϕ(x) is open wff with variable x, and perhaps others, free.
Gödel’s Speedup Theorem
Gödel’s Speedup Theorem

Let $i \geq 0$, and let $f$ be any recursive function.
Gödel’s Speedup Theorem

Let \( i \geq 0 \), and let \( f \) be any recursive function.

Then there is an infinite family \( \mathcal{F} \) of \( \Pi^0_1 \) formulae such that:
Gödel’s Speedup Theorem

Let $i \geq 0$, and let $f$ be any recursive function.
Then there is an infinite family $\mathcal{F}$ of $\Pi^0_1$ formulae such that:
Let $i \geq 0$, and let $f$ be any recursive function.

Then there is an infinite family $\mathcal{F}$ of $\Pi^0_1$ formulae such that:

1. $\forall \phi \in \mathcal{F}, Z_i \vdash \phi$; and
2. $\forall \phi \in \mathcal{F}$, if $k$ is the least integer s.t.
   $Z_{i+1} \vdash^k \text{symbols } \phi$, then $Z_i \nvdash f(k) \text{ symbols } \phi$. 

Gödel’s Speedup Theorem
Let $i \geq 0$, and let $f$ be any recursive function.

Then there is an infinite family $\mathcal{F}$ of $\Pi^0_1$ formulae such that:

1. $\forall \phi \in \mathcal{F}, Z_i \vdash \phi$; and

2. $\forall \phi \in \mathcal{F}$, if $k$ is the least integer s.t. $Z_{i+1} \vdash k$ symbols $\phi$, then $Z_i \not\vdash f(k)$ symbols $\phi$.

Can you build an AI that can prove this??
Gödel’s Speedup Theorem

Let $i \geq 0$, and let $f$ be any recursive function.

Then there is an infinite family $\mathcal{F}$ of $\Pi_1^0$ formulae such that:

1. $\forall \phi \in \mathcal{F}, Z_i \vdash \phi$; and
2. $\forall \phi \in \mathcal{F}$, if $k$ is the least integer s.t. $Z_{i+1} \vdash^k \phi$, then $Z_i \not\vdash^f(k) \phi$.

Can you build an AI that can prove this??

Yes. Somehow …
Supererogatory Reflection
Supererogatory Reflection

Prove $\vdash_{PA_1} 1000 \geq 0$; how long is your proof?
Supererogatory Reflection

Prove $\vdash_{\text{PA}_1} 1000 \geq 0$; how long is your proof?

Prove $\vdash_{\text{PA}_2} 1000 \geq 0$; how short can you make your proof?
And ... time travel ...
And ... time travel ...