Godel’s Ontological Argument

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Godel’s Great Theorems

- Introduction (“The Wager”)
- Brief Preliminaries (e.g. the propositional calculus & FOL)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
- The Continuum-Hypothesis Theorem
- The Time-Travel Theorem
- Gödel’s “God Theorem”
- Could a Finite Machine Match Gödel’s Greatness?

Background:
Godel and Religion

Kurt Godel

Born in 1906 in Brun, Austria-Hungary

Proved some of the most important theorems in logic.

Einsteins best friend 1940 - 1955

Widely considered to be the greatest modern logician: (Quotes from [Wang 1996])

- “No one denies that his position among logicians is comparable to Einstein's among physicists” - Hao Wang
- “If you called him the greatest logician since Aristotle you'd be downgrading him” -André Weil
- “Gödel was the only person who could speak without exaggeration of ‘Aristotle and me.’” -John Wheeler

Some Quotes On Godel's Religious Worldview

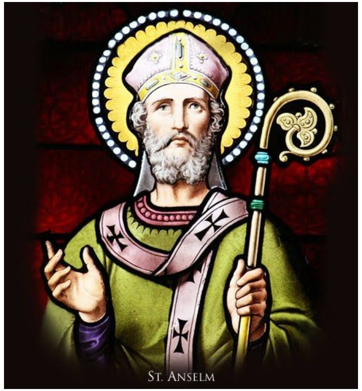
“Gödel gave his own religion as "baptized" Lutheran (though not a member of any religious congregation) and noted that his belief was theistic” [Wang 1996]

“In 1978 [Godel's wife] said that Gödel read the Bible in bed on Sundays although he did not go to church.” [Wang 1996]

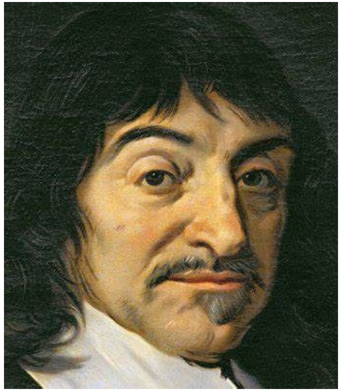
Godel’s Ontological Argument

Influences

Primary influence is Leibniz's ontological argument, which itself is made in response to Descartes ontological argument, which itself an independently discovered reformulation of St. Anselm's ontological argument:

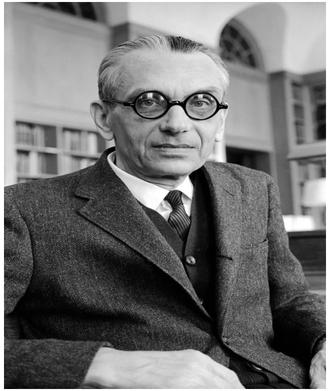


St. Anselm



Descartes

Leibniz



Godel

Influences: Leibniz

Leibniz's ontological argument can be read as follows:

(1) God is a being having all perfections. (Definition)
(2) A perfection is a simple and absolute property. (Definition)
(3) Existence is a perfection.
(4) If existence is part of the essence of a thing, then it is a necessary being.
(5) If it is possible for a necessary being to exist, then a necessary being does exist.
(6) It is possible for a being to have all perfections.
(7) Therefore, a necessary being (God) does exist.

A Sketch of Godel’s argument

Formalize a notion of positive properties (“morally aesthetic” properties).

Formalize a notion of god in terms of a being with all positive properties.

Formalize a notion of essences as the fundamental properties of objects.

Formalize a notion of necessary existence in terms of essences.

Show god necessarily exists.

Positive Properties

A positive property is a property that is “good” in an “morally aesthetic” sense.

For example, being good is a positive property, being knowledgeable is a positive property, being all knowing (omniscient) is a positive property.

“PositiveProp” is a 2nd order predicate, it is a property of a property.

“selmer” is an object	: object
“knowledgeable” is a property	: object -> Bool
“PositiveProp” is a property of a property	: (object -> Bool) -> Bool

Knowledgeable(selmer) is the statement “Selmer is Knowledgeable”

Positive(Knowledgeable) is the statement “Being Knowledgeable is a positive property”

Positive Properties : Axiom 1

For any property φ let P(φ) be read "φ is a positive property"

Axiom 1: Either a property φ or its negation $\neg\varphi$ is positive but not both.

$$\forall \varphi : P(\neg\varphi) \iff \neg P(\varphi)$$

Example: If being smart is a positive property then being dumb (not smart) can not be a positive property.

Exercises in HOL workspace in Hyperslate, prove Axiom 1 is equivalent to:

$$\forall \varphi : P(\varphi) \iff \neg P(\neg\varphi)$$

$$\forall \varphi : P(\varphi) \oplus P(\neg\varphi)$$

You may assume propositional extensionality:
$\forall a,b: (a \leftrightarrow b) \rightarrow (a = b)$

This XOR version is closer to the natural language reading of Ax1

Pop Quiz 1

Create a new Higher Order Logic workspace,

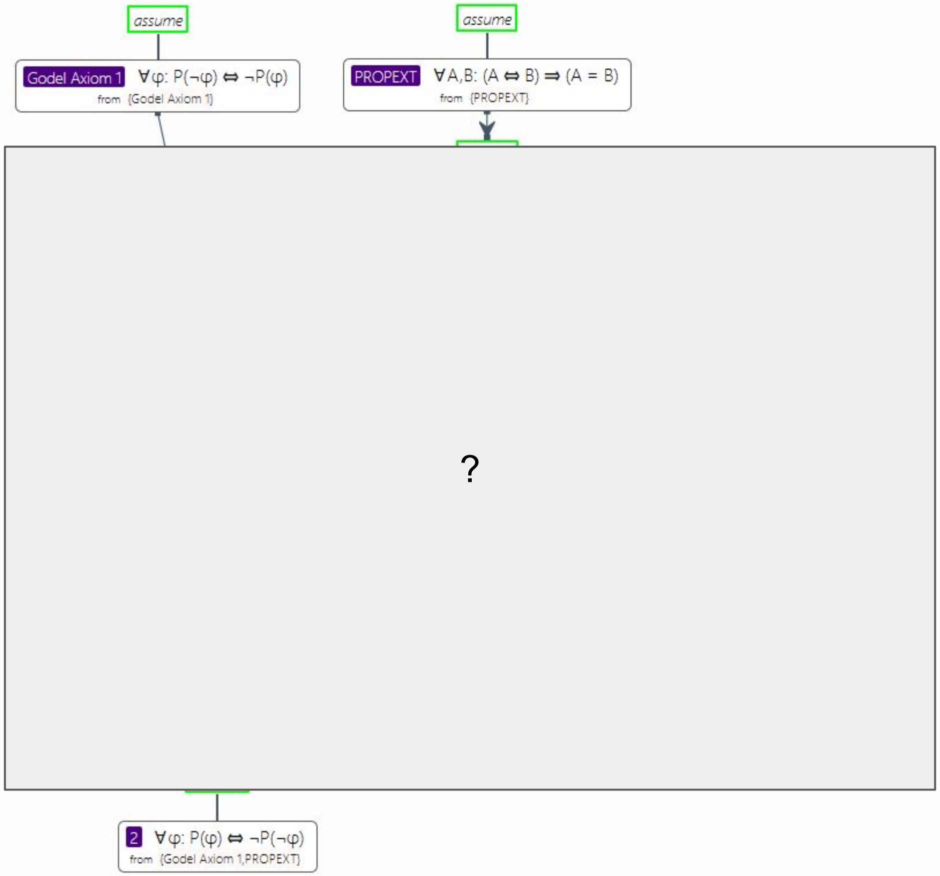
Prove from Ax 1:

Either a property φ or its negation $\neg\varphi$ is positive but not both.

$$\forall\varphi : P(\neg\varphi) \iff \neg P(\varphi)$$

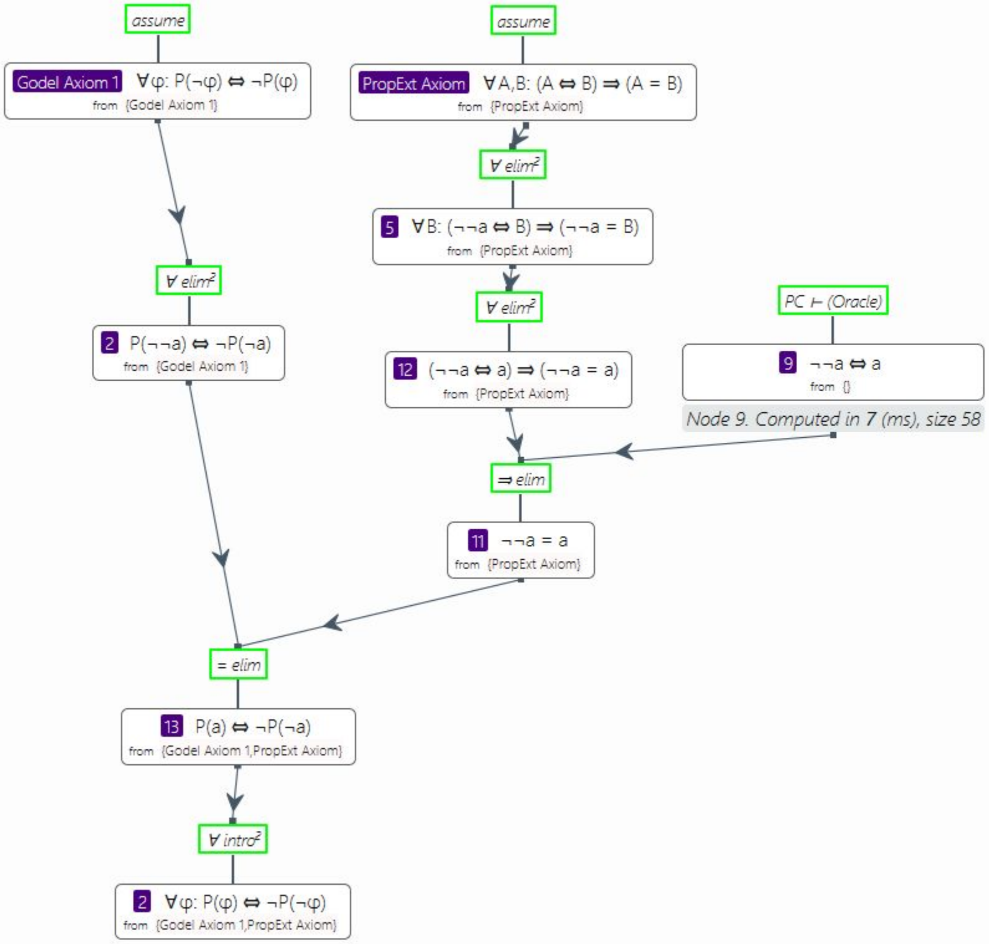
And propositional extensionality that:

$$\forall\varphi : P(\varphi) \iff \neg P(\neg\varphi)$$



A Solution

- Hinges on 2nd order forall elimination, replacing *φ* with the arbitrary formula ¬a.



Positive Properties : Axiom 2

Axiom 2: A property ψ necessarily implied by a positive property ϕ is positive.

$$\forall \phi, \psi : (P(\phi) \wedge \Box(\forall x : \phi(x) \rightarrow \psi(x))) \rightarrow P(\psi)$$

This axiom captures the notion that positive properties have ontological relations to other positive properties.

Ex.

Assume being huge is a possessive property, P(Huge).

It is necessary that anything huge is big. (In all possible worlds anything that huge is big)

Then we can derive that being big is a positive property, P(Big)

Positive Properties : Theorem 1

Theorem 1: Positive properties are possibly exemplified. I.E For a positive property φ it is possible that something exists with this property.

$$\forall\varphi : P(\varphi) \to \Diamond\exists x : \varphi(x)$$

Provable from axioms 1 and 2.

Fun exercise on paper (try a proof by contradiction).

The Definition of God and Godliness

God is an object that possesses all positive properties.

We will define *godliness* G as the property of being god, i.e. the property that and object possesses all positive properties.

Def 1: A Godly being x possesses all positive properties

$$G(x) := \forall\varphi : P(\varphi) \to \varphi(x)$$

Axiom About Godliness

Axiom 3: The property of being godly is a positive property.

$$P(G)$$

This axiom captures the fairly intuitive notion of that being godly is in and of itself a positive property.

As being godly by definition means you have all positive properties, it seems fair to say that the property of having all positive properties is a positive property.

It is possible God exists : Corollary 1

Corollary 1:

God possibly exists.

$$\Diamond \exists x : G(x)$$

This follows immediately from Theorem 1 and Axiom 3:

$$\begin{aligned} \forall \varphi : P(\varphi) \to \Diamond \exists x : \varphi(x) &\quad \text{(Theorem 1)} \\ P(G) \to \Diamond \exists x : G(x) &\quad (\forall_2 \text{ Elim}) \\ P(G) &\quad \text{(Axiom 3)} \\ \Diamond \exists x : G(x) &\quad (\to \text{ Elim}) \end{aligned}$$

If you accept all the axioms so far, then it is a theorem that god possibly exists.

Positive Properties : Axiom 4

Axiom 4: Positive properties are necessarily positive.

$$\forall\varphi : P(\varphi) \rightarrow \Box P(\varphi)$$

This axiom captures the notion that if something is a positive property, it is positive everywhere (in all possible worlds).

Ex. If being knowledgeable is a positive property then being knowledgeable is a positive property in all worlds.

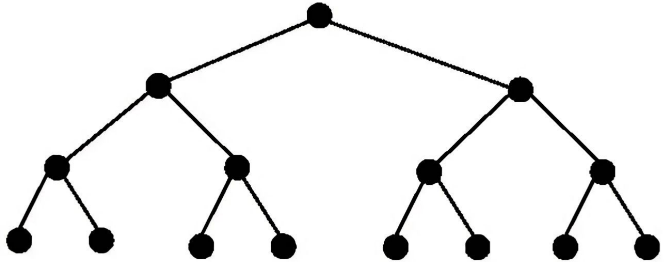
Essences: Definition 2

Definition 2: An essence of an individual x is a property it possesses that necessarily implies all of x's other properties.

$$Ess(\varphi, x) := \varphi(x) \wedge (\forall \psi : \psi(x) \rightarrow (\Box \forall y : \varphi(y) \rightarrow \psi(y)))$$

Type Signature of Ess : ((object -> Bool) x object) -> Bool

The notion of essence captures the natural language notion of essence, as a single property at the heart of something.



Essences : Theorem 2

Theorem 2: Being godly is the essence of a godly being.

$$\forall x : G(x) \to Ess(G, x)$$

Follows minimally from Ax 1, Def 1, Ax 4, Def 2

Informal argument:

If x is godly then it has all positive properties and no negative properties (Def 1). G thus implies all properties of x necessarily (Def 1, Ax 4). x has the property G and G implies all of its x's other properties necessarily, G is thus the essence of x (Def 2).

Necessary Existence : Definition 3

Definition 3:

Necessary existence of an individual x is defined as the necessary exemplification of it's essences. In other words, an individual necessarily exists if its essences have an object that embodies them in all possible worlds.

$$NE(x) := \forall\varphi : Ess(\varphi, x) \rightarrow \Box\exists y : \varphi(y)$$

Essences allow for the capture of necessary existence across worlds, as objects in different worlds can interpreted the same if they share all essences.

Necessary Existence: Axiom 5

Axiom 5: Necessary existence is a positive property.

$$P(NE)$$

Captures the notion that existence and more broadly a form of maximal existence is good.

It is Necessary God Exists

Theorem 3 It is necessary god exists.

$$\Box \exists x : G(x)$$

Follows from Def 1, Corollary 1, Thm 2, Def 3, Axiom 5.

Requires that in each world there exists a godly object x IE, an object with all positive properties.

Proof hinges on Leibniz's idea:

> If it is possible for a necessary being to exist, then a necessary being does exist.

A Monotheism Corollary

If we assume the *identity of indiscernibles*: that objects sharing the same properties are equal, even in different worlds,

Then…

Godel's god must be unique across all worlds, since the object in representing god in each world has the property G, which entails identical properties in all worlds.

Provides an argument that Godel's god is thus immutable and monotheistic.

Formal Verification

In 2014, Christoph Benzmuller and Bruno Paleo formally verified that the version of Godel's proof we have provided is valid with respect to the semantics of the higher order modal logic used for the proof. [Benzmuller 2014]

Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers

Christoph Benzmüller[1] and **Bruno Woltzenlogel Paleo**[2]

Abstract. Kurt Gödel's ontological argument for God's existence has been formalized and automated on a computer with higher-order automated theorem provers. From Gödel's premises, the computer proved: necessarily, there exists God. On the other hand, the theorem provers have also confirmed prominent criticism on Gödel's ontological argument, and they found some new results about it.

The background theory of the work presented here offers a novel perspective towards a *computational theoretical philosophy*.

1 INTRODUCTION

Kurt Gödel proposed an argumentation formalism to prove the existence of God [23, 30]. Attempts to prove the existence (or non-existence) of God by means of abstract, ontological arguments are an old tradition in western philosophy. Before Gödel, several prominent philosophers, including St. Anselm of Canterbury, Descartes and Leibniz, have presented similar arguments. Moreover, there is an impressive body of recent and ongoing work (cf. [31, 19, 18] and the references therein). Ontological arguments, for or against the existence of God, illustrate well an essential aspect of metaphysics: some (necessary) facts for our existing world are deduced by purely a priori, analytical means from some abstract definitions and axioms.

What motivated Gödel as a logician was the question, whether it is possible to deduce the existence of God from a small number of foundational (but debatable) axioms and definitions, with a mathe-

A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \equiv \neg P(\phi)]$

A2 A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \land \Box\forall x[\phi(x) \supset \psi(x)]) \supset P(\psi)]$

T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \supset \Diamond\exists x\phi(x)]$

D1 A *God-like* being possesses all positive properties: $G(x) \equiv \forall\phi[P(\phi) \supset \phi(x)]$

A3 The property of being God-like is positive: $P(G)$

C Possibly, God exists: $\Diamond\exists x G(x)$

A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \supset \Box P(\phi)]$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \equiv \phi(x) \land \forall\psi(\psi(x) \supset \Box\forall y(\phi(y) \supset \psi(y)))$

T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \supset G\ ess.\ x]$

D3 *Necessary existence* of an individ. is the necessary exemplification of all its essences: $NE(x) \equiv \forall\phi[\phi\ ess.\ x \supset \Box\exists y\phi(y)]$

A5 Necessary existence is a positive property: $P(NE)$

T3 Necessarily, God exists: $\Box\exists x G(x)$

Figure 1. Scott's version of Gödel's ontological argument [30].

Consistency

Recall that it is possible for systems of axioms to be *inconsistent*, that is, we can derive a contradiction from the axioms, and thus by explosion, prove anything.

Do we need to worry about that in this system?

No, the axioms and definitions are proven consistent by [Benzmuller 2014], we can not use them to derive a contradiction.

Criticism : Modal Collapse

The primary criticism of Godel's ontological argument is *modal collapse*.

Under Godel's axioms, the formula $\varphi \rightarrow \Box \varphi$ is a theorem

In their 2014 paper [Benzmuller 2014], Benzmuller and Paleo formally verified this. It is minimally provable from Def 2, Theorem 2, Theorem 3.

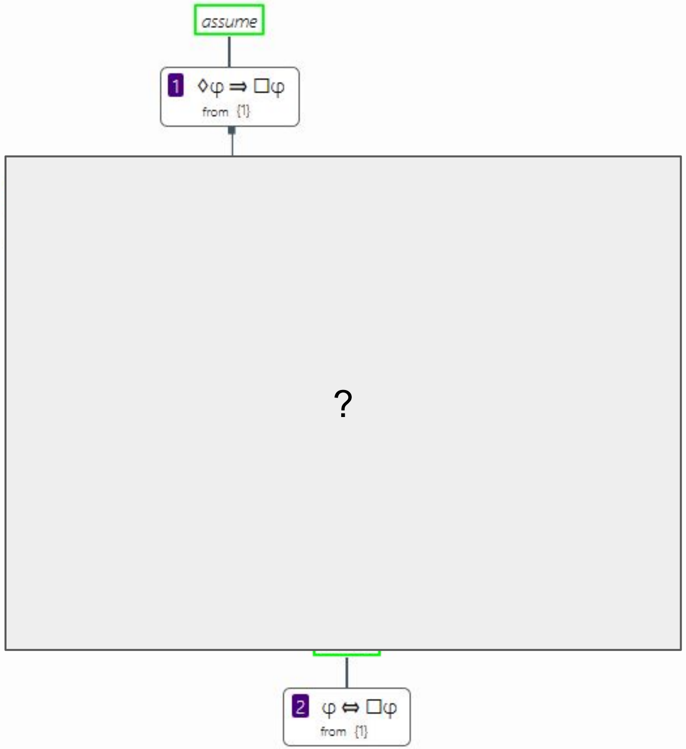
Modal collapse can cause large issues with reasoning as it means that anything is necessary.

Some theorize that this decision was intentional, or at least not seen as an issue, by Godel. [Koons 2005]

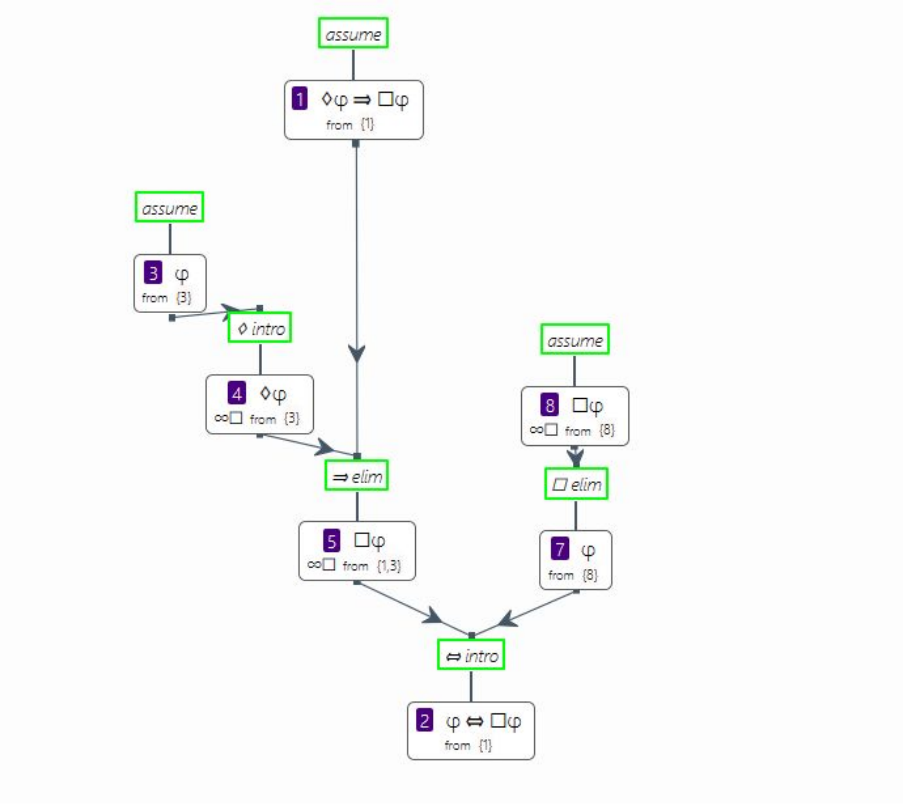
Pop Quiz : Modal Collapse

Create a new S5 workspace.

Prove that any situation in which something is possible implies something is necessary results in a modal collapse result for that statement, where we can introduce and remove boxes arbitrarily.



A Solution



assume
1 ◊φ ⇒ □φ
from {1}
assume
3 φ
from {3}
◊ intro
4 ◊φ
∞□ from {3}
⇒ elim
5 □φ
∞□ from {1,3}
assume
8 □φ
∞□ from {8}
□ elim
7 φ
from {8}
⇔ intro
2 φ ⇔ □φ
from {1}

Fin

Citations

[Wang 1996] A Logical Journey: From Gödel to Philosophy, https://doi.org/10.7551/mitpress/4321.001.0001

[Benzmuller 2014] Automating Gödel's ontological proof of God's existence with higher-order automated theorem provers, https://doi.org/10.3233/978-1-61499-419-0-93

[Koons 2005] Sobel on Godel’s Ontological Proof, https://web.archive.org/web/20200802202333/http://www.robkoons.net/media/69b0dd04a9d2fc6dffff80b4ffffd524.pdf