Addenda to Prop Calc;
The Case of Linda, & Probability, etc.

Atriya Sen

Are Humans Rational?

9/24/18

Selmer.Bringsjord@gmail.com
Addenda to Prop. Calc. …
Variables & Connectives
Variables & Connectives

Variables to represent declarative statements.
Variables & Connectives

Variables to represent declarative statements.

E.g., $k$ to represent ‘There is a king in the hand’.
Variables & Connectives

Variables to represent declarative statements.

E.g., $k$ to represent ‘There is a king in the hand’.

And five simple Boolean connectives (now, with the truth tables that define them):
Variables & Connectives

Variables to represent declarative statements.

E.g., $k$ to represent ‘There is a king in the hand’.

And five simple Boolean connectives (now, with the truth tables that define them):

not $\neg$  and $\land$  or (inclusive) $\lor$  if ... then ... $\rightarrow$  ... if and only if ... $\leftrightarrow$
**Variables & Connectives**

Variables to represent declarative statements.

E.g., $k$ to represent ‘There is a king in the hand’.

And five simple Boolean connectives (now, with the truth tables that define them):


<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\neg \phi$</th>
<th>$\phi \land \psi$</th>
<th>$\phi \lor \psi$</th>
<th>$\phi \rightarrow \psi$</th>
<th>$\phi \leftrightarrow \psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>


And a Crucial Concept & Its Notation: *Provability*
And a Crucial Concept & Its Notation: **Provability**

\[ \Phi \vdash \_{PC} \phi \]
And a Crucial Concept & Its Notation: **Provability**

\[ \Phi \vdash_{PC} \phi \]

set of formulae in the propositional calculus
And a Crucial Concept & Its Notation: **Provability**

\[ \Phi \vdash_{PC} \phi \]

- **\( \Phi \)**: set of formulae in the propositional calculus
- **\( \vdash_{PC} \)**: indicates that the formula on the right can be proved from the set of formulae on the left, using inference rules in PC (prop. calc.).
And a Crucial Concept & Its Notation: **Provability**

\[ \Phi \vdash_{PC} \phi \]

- **Set of formulae in the propositional calculus**
- **Individual formula in the propositional calculus**

This notation indicates that the formula on the right can be proved from the set of formulae on the left, using inference rules in PC (prop. calc.).
And a Crucial Concept & Its Notation: **Provability**

\[ \Phi \vdash_{PC} \phi \]

- **set of formulae in the propositional calculus**
- **indicates that the formula on the right can be proved from the set of formulae on the left, using inference rules in PC (prop. calc.).**
- **individual formula in the propositional calculus**

So, the equation reads like this:
And a Crucial Concept & Its Notation: **Provability**

\[ \Phi \vdash_{PC} \phi \]

- set of formulae in the propositional calculus
- indicates that the formula on the right can be proved from the set of formulae on the left, using inference rules in PC (prop. calc.).
- individual formula in the propositional calculus

So, the equation reads like this:

“Formula $\phi$ is provable in the propositional calculus from the set $\Phi$ of formula.”
And a Crucial Concept & Its Notation: 

**Provability**

\[ \Phi \vdash_{PC} \phi \]

- \( \Phi \): set of formulae in the propositional calculus
- \( \vdash_{PC} \): indicates that the formula on the right can be proved from the set of formulae on the left, using inference rules in PC (prop. calc.)
- \( \phi \): individual formula in the propositional calculus

So, the equation reads like this:

“Formula \( \phi \) is provable in the propositional calculus from the set \( \Phi \) of formula.”

\( \vdash \): simply means, then, that \( \phi \) can be proved with only temporary suppositions.
Selmer’s Monty Fall Problem …
Jones has come to a game show, and finds himself thereon selected to play a game on national TV with the show's suave host, Full Monty. Jones is told correctly by Full that hidden behind one of three closed, opaque doors facing the two of them is $1,000,000, while behind each of the other two is a feculent, obstreperous llama whose value on the open market is charitably pegged at $1. Full reminds Jones that this is a game, and a fair one, and that if Jones ends up selecting the door with $1M behind it, all that money will indeed be his. (Jones' net worth has nearly been exhausted by his expenditures in traveling to the show.)

But, sometimes a mad professor of probability who is in the audience disguised jumps up and shouts out which number, 1, 2, or 3, his random number generator has just generated, and the door number he shouts out immediately thereafter opens. If the door the contestant has picked opens, the contestant gets what has been revealed. But, if one of the other two doors opens, the contestant is allowed to switch by Full Monty. All of this was explained to Jones before the game began.

Full asks Jones to select which door he wants the contents of. Jones says, "Door 1." Full then says: "Okay. Now let's op—"

Suddenly the mad professor jumps up and shouts out "2!" and immediately thereafter Door 2 opens to reveal a llama.

(P2.1) What should Jones do if he's rational?

(P2.2) Prove that your answer is correct. (Diagrammatic proofs are allowed.)
So what’s got to be in that mind to make it rational?!?

Success v Failure

\(< \theta, \pi > \rightarrow_{\text{Li}} \text{“proof”} \)
So what’s got to be in that mind to make it rational?!?

Success v Failure

A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
So what’s got to be in that mind to make it rational?!?

Success v Failure

\(< \varphi, \pi > \rightarrow_{Li} \text{“proof”}\)

A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
So what’s got to be in that mind to make it rational?!?

\[ <\varnothing, \pi> \rightarrow_{\text{Li}} \text{“proof”} \]

Success v Failure

A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
15

Linda: Less Is More

The best-known and most controversial of our experiments involved a fictitious lady called Linda. Amos and I made up the Linda problem to provide conclusive evidence of the role of heuristics in judgment and of their incompatibility with logic. This is how we described Linda:

Linda is thirty-one years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations.

The audiences who heard this description in the
By the way, as mentioned, the “crisis” roiling around priming:

https://replicationindex.wordpress.com/2017/02/02/reconstruction-of-a-train-wreck-how-priming-research-went-of-the-rails/#comment-1454

Kahneman: “… I placed too much faith in underpowered studies …”
1980s always laughed because they immediately knew that Linda had attended the University of California at Berkeley, which was famous at the time for its radical, politically engaged students. In one of our experiments we presented participants with a list of eight possible scenarios for Linda. As in the Tom W problem, some ranked the scenarios by representativeness, others by probability. The Linda problem is similar, but with a twist.

Linda is a teacher in elementary school.
Linda works in a bookstore and takes yoga classes.
Linda is active in the feminist movement.
Linda is a psychiatric social worker.
Linda is a member of the League of Women Voters.
Linda is a bank teller.
Linda is an insurance salesperson.

Linda is a bank teller and is active in the feminist movement.

The problem shows its age in several ways. The League of Women Voters is no longer as prominent as it was, and the idea of a feminist “movement” sounds quaint, a testimonial to the change in the status of women over the last thirty years. Even in the Facebook era, however, it is still easy to guess the almost perfect consensus of judgments: Linda is a very good fit for an active feminist, a fairly good fit for someone who works in a bookstore and takes yoga classes—and a very poor fit for a bank teller or an insurance salesperson.

Now focus on the critical items in the list: Does Linda look more like a bank teller, or more like a bank teller who is active in the feminist movement? Everyone agrees that Linda fits the idea of a “feminist bank teller” better than she fits the stereotype
of bank tellers. The stereotypical bank teller is not a feminist activist, and adding that detail to the description makes for a more coherent story.

The twist comes in the judgments of likelihood, because there is a logical relation between the two scenarios. Think in terms of Venn diagrams. The set of feminist bank tellers is wholly included in the set of bank tellers, as every feminist bank teller is a bank teller. Therefore the probability that Linda is a feminist bank teller must be lower than the probability of her being a bank teller. When you specify a possible event in greater detail you can only lower its probability. The problem therefore sets up a conflict between the intuition of representativeness and the logic of probability.

Our initial experiment was between-subjects. Each participant saw a set of seven outcomes that included only one of the critical items ("bank teller" or "feminist bank teller"). Some ranked the outcomes by resemblance, others by likelihood. As in the case of Tom W, the average rankings by resemblance and by likelihood were identical; "feminist bank teller" ranked higher than "bank teller" in both.

Then we took the experiment further, using a within-subject design. We made up the questionnaire as you saw it, with "bank teller" in the sixth position in the list and "feminist bank teller" as the last item. We were convinced that subjects would notice the relation between the two outcomes, and that their rankings would be consistent with logic. Indeed, we were so certain of this that we did not think it worthwhile to conduct a special experiment. My assistant was running another experiment in the lab, and she asked the subjects to complete the new Linda questionnaire while signing out, just before they got paid.

About ten questionnaires had accumulated in a
tray on my assistant’s desk before I casually glanced at them and found that all the subjects had ranked “feminist bank teller” as more probable than “bank teller.” I was so surprised that I still retain a “flash-bulb memory” of the gray color of the metal desk and of where everyone was when I made that discovery. I quickly called Amos in great excitement to tell him what we had found: we had pitted logic against representativeness, and representativeness had won!

In the language of this book, we had observed a failure of System 2: our participants had a fair opportunity to detect the relevance of the logical rule, since both outcomes were included in the same ranking. They did not take advantage of that opportunity. When we extended the experiment, we found that 89% of the undergraduates in our sample violated the logic of probability. We were convinced that statistically sophisticated respondents would do better, so we administered the same questionnaire to doctoral students in the decision-science program of the Stanford Graduate School of Business, all of whom had taken several advanced courses in probability, statistics, and decision theory. We were surprised again: 85% of these respondents also ranked “feminist bank teller” as more likely than “bank teller.”

In what we later described as “increasingly desperate” attempts to eliminate the error, we introduced large groups of people to Linda and asked them this simple question:

Which alternative is more probable?
Linda is a bank teller.
Linda is a bank teller and is active in the feminist movement.

This stark version of the problem made Linda
famous in some circles, and it earned us years of controversy. About 85% to 90% of undergraduates at several major universities chose the second option, contrary to logic. Remarkably, the sinners seemed to have no shame. When I asked my large undergraduate class in some indignation, “Do you realize that you have violated an elementary logical rule?” someone in the back row shouted, “So what?” and a graduate student who made the same error explained herself by saying, “I thought you just asked for my opinion.”

The word fallacy is used, in general, when people fail to apply a logical rule that is obviously relevant. Amos and I introduced the idea of a conjunction fallacy, which people commit when they judge a conjunction of two events (here, bank teller and feminist) to be more probable than one of the events (bank teller) in a direct comparison.

As in the Müller-Lyer illusion, the fallacy remains attractive even when you recognize it for what it is. The naturalist Stephen Jay Gould described his own struggle with the Linda problem. He knew the correct answer, of course, and yet, he wrote, “a little homunculus in my head continues to jump up and down, shouting at me—‘but she can’t just be a bank teller; read the description.’” The little homunculus is of course Gould’s System 1 speaking to him in insistent tones. (The two-system terminology had not yet been introduced when he wrote.)

The correct answer to the short version of the Linda problem was the majority response in only one of our studies: 64% of a group of graduate students in the social sciences at Stanford and at Berkeley correctly judged “feminist bank teller” to be less probable than “bank teller.” In the original version with eight outcomes (shown above), only 15% of a similar group of graduate students had
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
Urn-based Reasoning

A1: Deductive Tools
A2: Inductive Tools
A3: Analysis
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis

Urn-based Reasoning
Kolmogorov’s Axioms
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis

Urn-based Reasoning
Kolmogorov’s Axioms
Elementary Probability Logic
(based on propositional calculus)
A1: Deductive Tools
A2: Inductive Tools
A3: Analysis

Urn-based Reasoning
Kolmogorov’s Axioms
Elementary Probability Logic
(based on propositional calculus)
“Narratological” Probability

to Defend the Subjects
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball \( b_n \) is a black ball?

What is the probability that (B) \( b_3 \) is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?
Probability of drawing a black ball?

Probability that ball $b_n$ is a black ball?

What is the probability that (B) $b_3$ is a black ball?

$p(B) = \frac{4}{5} = .8.$
What are you more likely to pull out, a BT *simpliciter*, or a BT-and-FM?
What are you more likely to pull out, a BT *simpliciter*, or a BT-and-FM?

- 1 BT-and-FM
- 4 BTs
- $k$ people
What are you more likely to pull out, a BT *simpliciter*, or a BT-and-FM?
What are you more likely to pull out, a BT simpliciter, or a BT-and-FM?

\[ \theta, \pi \rightarrow_{\mu} \text{prop!} \]
What are you more likely to pull out, a BT *simpliciter*, or a BT-and-FM?
What are you more likely to pull out, a BT *simpliciter*, or a BT-and-FM?
What are you more likely to pull out, a BT *simpliciter*, or a BT-and-FM?

A BT!
Kolmogorov’s Axioms of Probability

K1 \( \forall \phi (0 \leq p(\phi) \leq 1) \).
Each formula in the propositional calculus has a probability between 0 and 1, inclusive.

K2 If \( \vdash \phi \), then \( p(\phi) = 1 \).
All formulas that are deductively provable without remaining suppositions are certain.

K3 If \( \{\phi\} \vdash \psi \), then \( p(\phi) \leq p(\psi) \).
A formula that can be used to prove another has a probability less than or equal to the proved one.

K4 If \( \{\phi, \psi\} \vdash \delta \land \neg \delta \), then \( p(\phi \lor \psi) = p(\phi) + p(\psi) \).
Two inconsistent formulas, disjoined, have a probability equal to the sum of the probability of each.
Kolmogorov’s Axioms of Probability

**K1** \( \forall \phi (0 \leq p(\phi) \leq 1) \).
Each formula in the propositional calculus has a probability between 0 and 1, inclusive.

**K2** If \( \vdash \phi \), then \( p(\phi) = 1 \).
All formulas that are deductively provable without remaining suppositions are certain.

**K3** If \( \{ \phi \} \vdash \psi \), then \( p(\phi) \leq p(\psi) \).
A formula that can be used to prove another has a probability less than or equal to the proved one.

**K4** If \( \{ \phi, \psi \} \vdash \delta \land \neg \delta \), then \( p(\phi \lor \psi) = p(\phi) + p(\psi) \).
Two inconsistent formulas, disjoined, have a probability equal to the sum of the probability of each.

So why is Kahneman right that System-2 cognition tells us that (B and F) cannot be more probable than B???
Kolmogorov’s Axioms of Probability

K1  \( \forall \phi (0 \leq p(\phi) \leq 1) \).
Each formula in the propositional calculus has a probability between 0 and 1, inclusive.

K2  If \( \vdash \phi \), then \( p(\phi) = 1 \).
All formulas that are deductively provable without remaining suppositions are certain.

K3  If \( \{\phi\} \vdash \psi \), then \( p(\phi) \leq p(\psi) \).
A formula that can be used to prove another has a probability less than or equal to the proved one.

K4  If \( \{\phi, \psi\} \vdash \delta \land \neg \delta \), then \( p(\phi \lor \psi) = p(\phi) + p(\psi) \).
Two inconsistent formulas, disjoined, have a probability equal to the sum of the probability of each.

So why is Kahneman right that System-2
cognition tells us that (B and F) cannot be
more probable than B???

Because from a conjunction \( \phi \land \psi \) of two formulas one can always prove \( \phi \) (and \( \psi \) as well).
Kolmogorov’s Axioms of Probability

**K1** \( \forall \phi (0 \leq p(\phi) \leq 1) \).
Each formula in the propositional calculus has a probability between 0 and 1, inclusive.

**K2** If \( \vdash \phi \), then \( p(\phi) = 1 \).
All formulas that are deductively provable without remaining suppositions are certain.

**K3** If \( \{\phi\} \vdash \psi \), then \( p(\phi) \leq p(\psi) \).
A formula that can be used to prove another has a probability less than or equal to the proved one.

**K4** If \( \{\phi, \psi\} \vdash \delta \land \neg \delta \), then \( p(\phi \lor \psi) = p(\phi) + p(\psi) \).
Two inconsistent formulas, disjoined, have a probability equal to the sum of the probability of each.

---

So why is Kahneman right that System-2 cognition tells us that (B and F) cannot be more probable than B???

Because from a conjunction \( \phi \land \psi \) of two formulas one can always prove \( \phi \) (and \( \psi \) as well).

Hence by **K3** it can *never* be the case that a conjunction is more probable than either of its conjuncts.
Kolmogorov’s Axioms of Probability

\[ \forall \phi (0 \leq p(\phi) \leq 1). \]
Each formula in the propositional calculus has a probability between 0 and 1, inclusive.
\[ \text{K2} \quad \text{If } \vdash \phi, \text{ then } p(\phi) = 1. \]
All formulas that are deductively provable without remaining suppositions are certain.
\[ \text{K3} \quad \text{If } \{\phi\} \vdash \psi, \text{ then } p(\phi) \leq p(\psi). \]
A formula that can be used to prove another has a probability less than or equal to the proved one.
\[ \text{K4} \quad \text{If } \{\phi, \psi\} \vdash \delta \land \neg \delta, \text{ then } p(\phi \lor \psi) = p(\phi) + p(\psi). \]
Two inconsistent formulas, disjoined, have a probability equal to the sum of the probability of each.

So why is Kahneman right that System-2
cognition tells us that (B and F) cannot be
more probable than B???

Because from a conjunction \( \phi \land \psi \) of two formulas one can always prove \( \phi \) (and \( \psi \) as well).
Hence by \text{K3} it can \textit{never} be the case that a conjunction is more probable than either of its conjuncts.

Hence it can never be the case that ‘Linda is a bank teller and Linda is
in the feminist movement’ is more probable than ‘Linda is a bank teller.’
Geometric Interpretation of Probability Logic
Geometric Interpretation of Probability Logic

![Venn Diagram with regions labeled B, F, 1, 2, 3, 4]
Geometric Interpretation of Probability Logic

Assume that the areas of the regions in the diagram represent the probabilities of the formulae they correspond to.
Assume that the areas of the regions in the diagram represent the probabilities of the formulae they correspond to.

Very well. Then why is it true that:
Assume that the areas of the regions in the diagram represent the probabilities of the formulae they correspond to.

Very well. Then why is it true that: \( p(B) > p(B \land F) \)
What’s really going on with Linda …
“Narratological” Probability
“Narratological” Probability
“Narratological” Probability
During college, math major Bruno was a stellar athlete, graduating Phi Beta Kappa from Princeton, where he received the Most Dedicated Athlete in his graduating class.
“Narratological” Probability

During college, math major Bruno was a stellar athlete, graduating Phi Beta Kappa from Princeton, where he received the Most Dedicated Athlete in his graduating class.
During college, math major Bruno was a stellar athlete, graduating Phi Beta Kappa from Princeton, where he received the Most Dedicated Athlete in his graduating class.

A: Bruno is a trauma surgeon.
“Narratological” Probability

During college, math major Bruno was a stellar athlete, graduating Phi Beta Kappa from Princeton, where he received the Most Dedicated Athlete in his graduating class.

A: Bruno is a trauma surgeon.

B: In leisure time, to relax, Bruno plays competitive tennis.
“Narratological” Probability

During college, math major Bruno was a stellar athlete, graduating Phi Beta Kappa from Princeton, where he received the Most Dedicated Athlete in his graduating class.

A: Bruno is a trauma surgeon.

B: In leisure time, to relax, Bruno plays competitive tennis.

Which is more probable [ ]? A alone or A and B together?
“Narratological” Probability

During college, math major Bruno was a stellar athlete, graduating Phi Beta Kappa from Princeton, where he received the Most Dedicated Athlete in his graduating class.

**A:** Bruno is a trauma surgeon.

**B:** In leisure time, to relax, Bruno plays competitive tennis.

Which is more probable [as a heading-toward-3D-picture of a person]? **A** alone or **A and B** together?
LABE Formalization

\[ S \vdash_{c.} \phi, S \vdash_{p} L, L \cup \{\beta\} \vdash_{\sigma \geq c.} \phi \]

\[ S \vdash_{\sigma \geq c.} \phi \land \beta \]