Rationality & Paradox, Part I:

The Liar;
The Barber;
(The Knowability Paradox);
& a Real-Life Paradox at Altitude …

Selmer Bringsjord

Are Humans Rational?

10/21/19

RAIR
Rensselaer AI and Reasoning Lab

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Types of Paradoxes

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- Inductive Paradoxes — corresponding to Area 2

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Today; Part I (& then Oct 28, Oct 31, Nov

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The Liar (Paradox) …

First, …
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(Why?)

The (Economical) Liar

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Contradiction!

The “Gödelian” Liar

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$\bar{P}$: This sentence is unprovable.

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Suppose that $\bar{P}$ is true. Then we can immediately deduce that $\bar{P}$ is provable (since from the supposition that some statement S holds we can prove $S \rightarrow S$ and then promptly deduce S by *modus ponens*). But since what $\bar{P}$ says is that it's unprovable, we have that $\bar{P}$ is false under our supposition.

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Suppose on the other hand that $\bar{P}$ is false. Then we can immediately deduce that $\bar{P}$ is unprovable (since if $\bar{P}$ were provable it would be true, because what it says is that it’s unprovable and we would have proved what it says). But since what $\bar{P}$ says is that it’s unprovable, and we proved $\bar{P}$ under our supposition, $\bar{P}$ is true.

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The (Verbose) Liar — With a Twist

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Proof: Set:

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L is either true or false. Suppose that it's true. Then since what it says is that it's false, it *is* false; i.e., **L** is false, on this supposition. So we've proved that if **L** is true, L is false. Now suppose instead that **L** is false. Then since it says that it's false, it's true; i.e., **L** is true, on our current supposition. We have thus proved that if **L** is false, **L** is true. Combining the conditionals we've proved yields this: **L** is true if and only if **L** is false, which is a contradiction. (*P* if and only if ¬*P* is logically equivalent to *P* and ¬*P*.) By the rule of inference *explosion*, it follows that 2+2 = 5. **QED**

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Box 1

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Box 2

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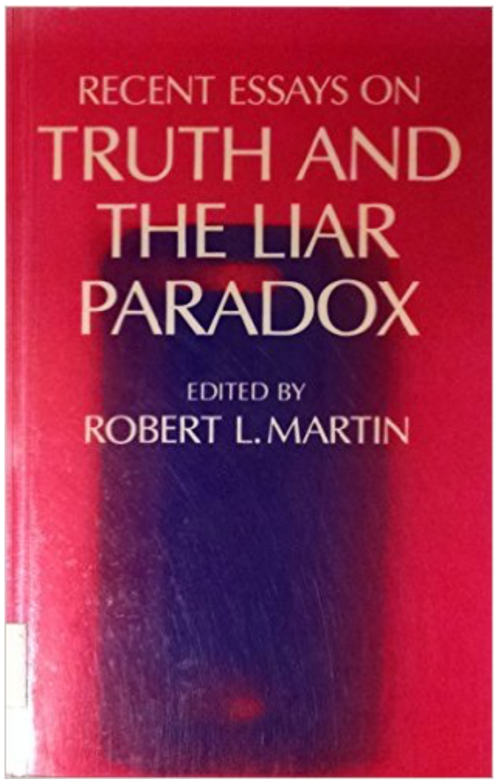
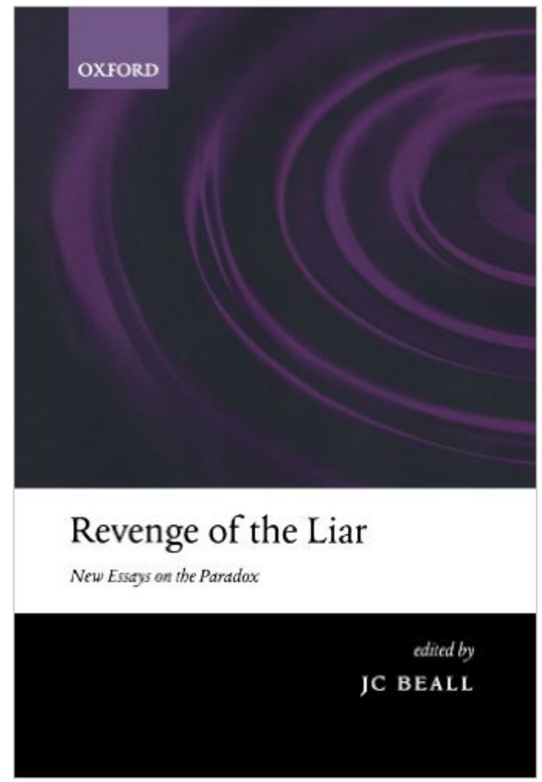
Suppose that the sentence in Box 1 is true. Then the sentence in Box 2 is true (because the sentence in Box 1 says that that sentence is true). But then the sentence in Box 1 is false (because the sentence in Box 2 says that that sentence is false). So, if the sentence in Box 1 is true, it's false. On the other hand, if the sentence in Box 1 is false, the sentence in Box 2 is false. But then the sentence in Box 1 is true; so we've shown that if the sentence in Box 1 is false, it's true. We thus have again a contradiction: The sentence in Box 1 is true if and only if it's not true!

Further Reading …





The Barber Paradox …

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Proof: Let's assume for the sake of argument that such a situation can be. Without loss of generality, let the town be Lyngdal and the male Lyngdalian barber be Olaf. Either Olaf shaves himself or he doesn't. But either case leads straight to a contradiction. Therefore the situation is in fact impossible. Here we go …

Suppose Olaf shaves himself. Then it follows that he doesn't shave himself. Suppose on the other hand that Olaf doesn't shave himself. Then is follows that he does shave himself. Hence, Olaf shaves himself if and only if he doesn't shave himself, which is a contradiction. **QED**

For homework …

For Presently Rational System-2ers (1/2)

- **Argument**: (1) Harrison admires only great actors who don't admire themselves. (2) Harrison admires all great actors who don't admire themselves. Therefore, (3) Harrison isn't a great actor.
- **Purported Proof to Certify the Argument**: For indirect proof, suppose that H is a great actor. Either H admires himself, or he doesn't. We show that either case leads to a contradiction, and hence that our starting supposition is false, i.e. H isn't a great actor. First, suppose that H does admire himself. From this and (1) we deduce that H doesn't admire himself — contradiction. For the second case, suppose that H doesn't admire himself. From this and (2) it follows that Harrison does admire himself — contradiction. Either alternative needs to the desired contradiction. **QED**
- Is the purported proof in fact a *valid* proof?

From Language, Proof, and Logic (2000) by Barwise & Etchemendy, p. 335.

For homework …

For Presently Rational System-2ers (2/2)

Show that it would be impossible to construct a reference book that lists all and only those reference books that do not list themselves.

From Language, Proof, and Logic (2000) by Barwise & Etchemendy, p. 337.

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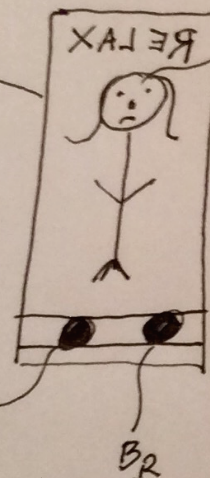
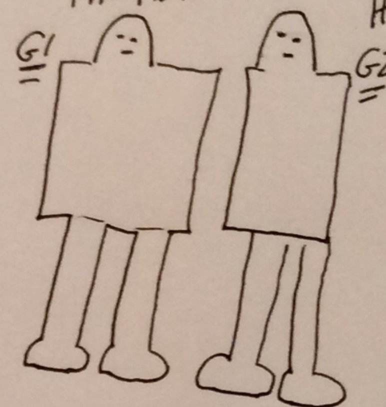
Dr Who and …
The Problem of the Osirians …

THE PROBLEM OF THE OSIRIANS

ONE QUESTION ALLOWED!

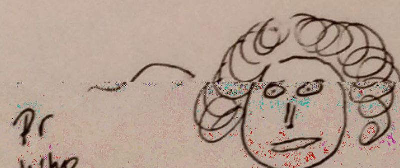
The Twin Guardians of Horus





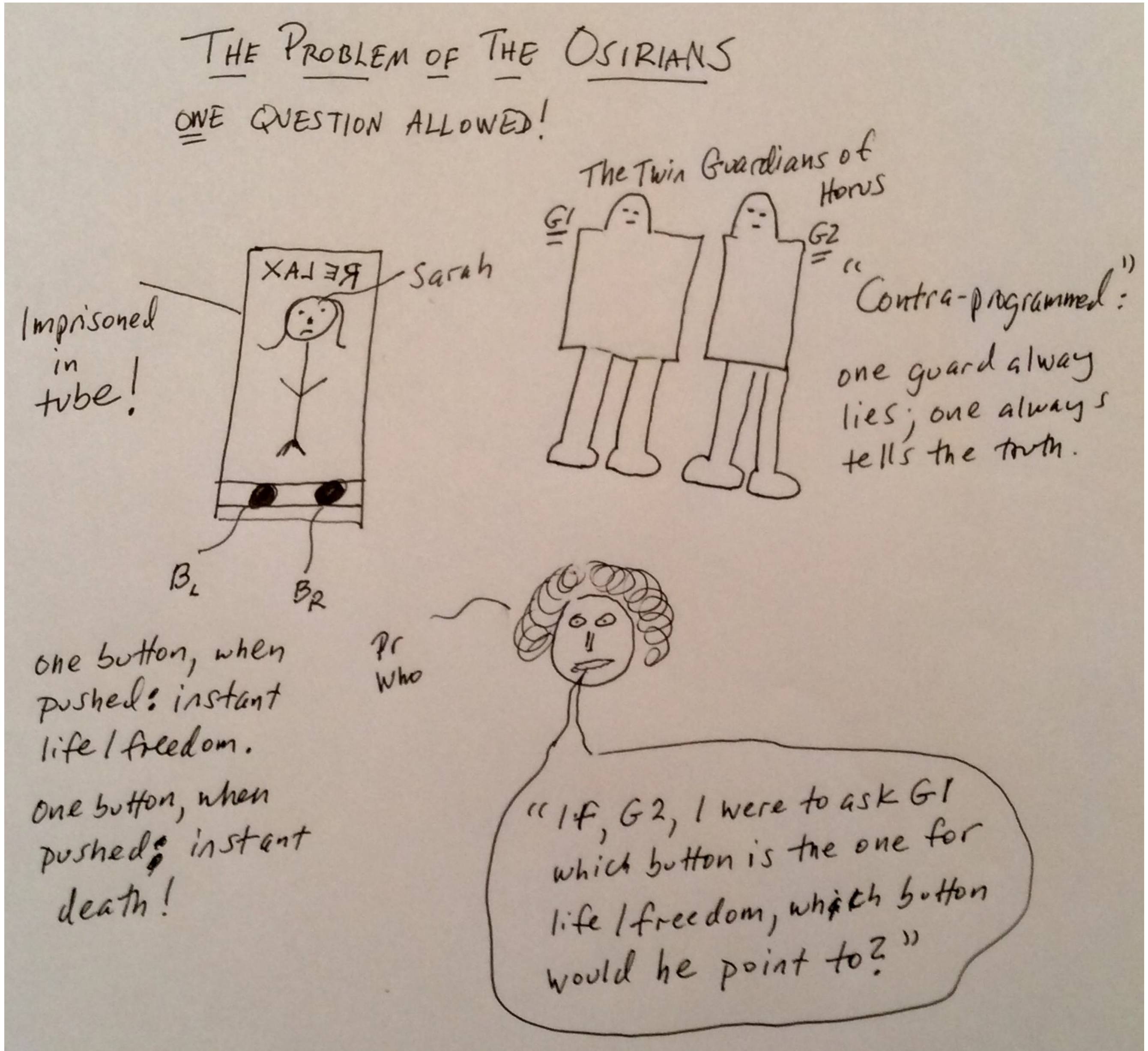
"Contra-programmed":

one guard always lies; one always tells the truth.



one button, when pushed: instant life/freedom.

one button, when pushed: instant death!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G2
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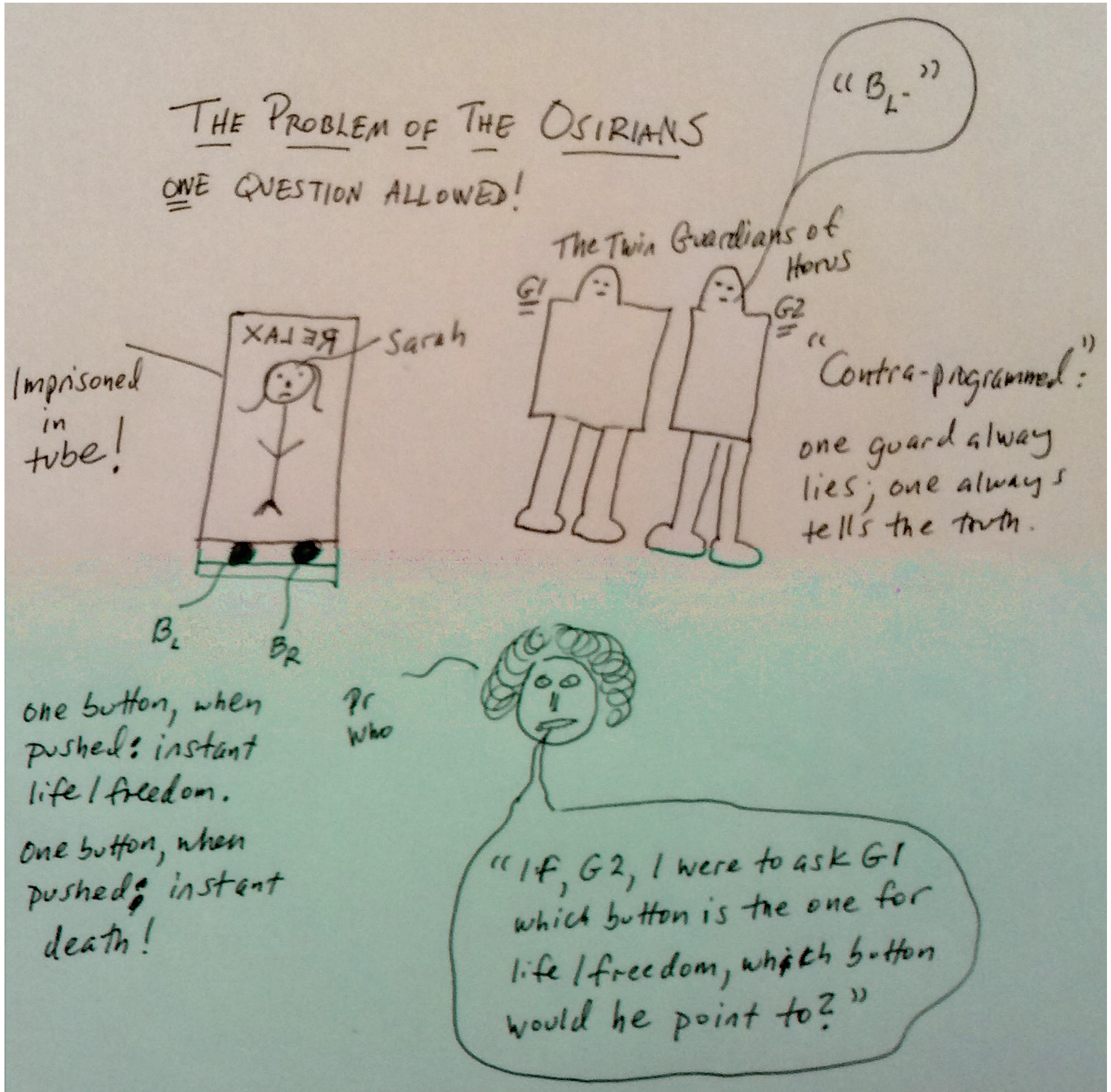
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Dr Who's (Background) Liar-Leveraging Proof

Proposition: B_R leads to freedom and life!

Proof: G2, who has uttered "B_L," is either a liar or a truth-teller. Suppose, first, the former case. Then G1 would in fact say "B_R" and would be telling the truth in so saying; hence B_R on this first supposition is the way to go. What about the latter case? In this case, G2 is telling the truth, and hence G1 would in fact say "B_L" — but would be lying, so in this case B_R is again the way to go. **QED**

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Syllabus:Schedule:December 2!

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Syllabus:Schedule:December 2!

Study the “Gödelian” Liar given earlier in this deck.

For Further (Supererogatory) Study: The Knowability Paradox …

Don't blame me. The true source only recently discovered in 2009: Alonzo Church, Turing's PhD advisor in the States, and the inventor of the lambda-calculus.

The Knowability Paradox (informal)

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Suppose that we know that there is at least one unknown truth; let's call it p*; this is an entirely arbitrary name. (While this is a supposition, it would be a very plausible assertion. After all, we're finite, non-omniscient beings. And don't we have examples?)

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Obviously, knowledge distributes over conjunctions. After all, if someone at some time knows that snow is white and that Tromsø is in Norway, it logically follows that someone at some time knows that snow is white, and someone at some time knows that Tromsø is in Norway. Hence:

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$$\mathbf{K}p^* \mathbin{\&} \mathbf{K}\neg\mathbf{K}p^* \qquad (2)$$

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Next, if someone at some time knows a proposition *s*, then *s* holds. We can apply this to the right conjunct in (2) to obtain:

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But (3) is a contradiction. Hence by *reductio ad absurdum*, we reject the original supposition that led to the contradiction. I.e., we deduce:

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What (6) says is this: There is an *unknowable* truth! — for what (6) declares is this: "It's impossible that someone at some time knows that the conjunction of p* & ¬Kp* holds." !

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Why is this (at least according to some) a paradox??

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Because it's supposed to be absurd that sitting in our armchairs, starting from the seemingly innocent and obvious fact that — and this is represented by (1) — there's at least one unknown truth, we can deduce that — and this is (6) — there's a truth that is absolutely unknow*able*.

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It's not possible that: **K**(p* & ¬**K**p*) (6)

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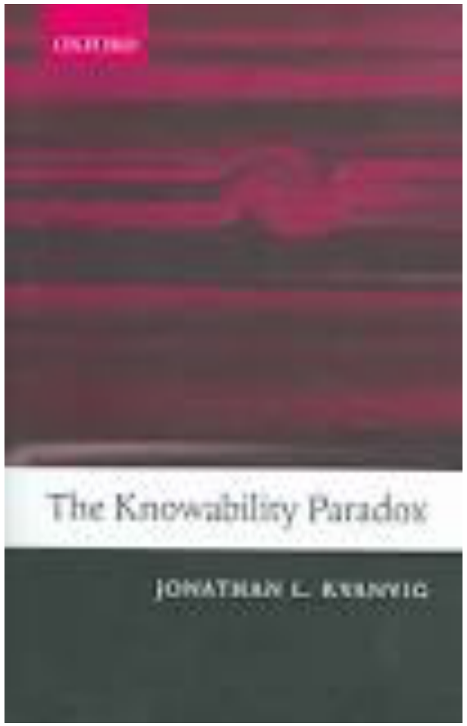
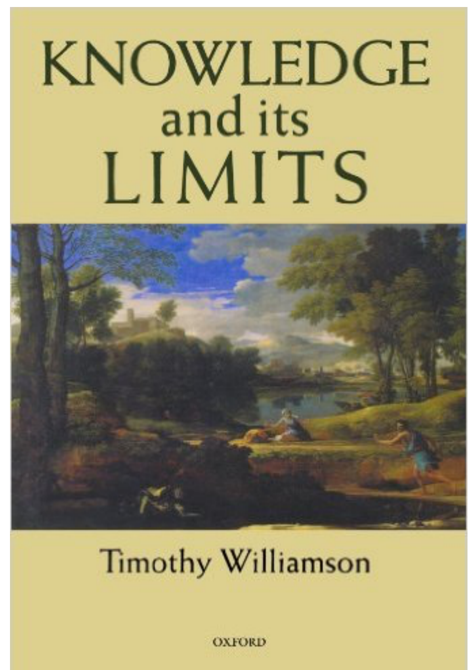
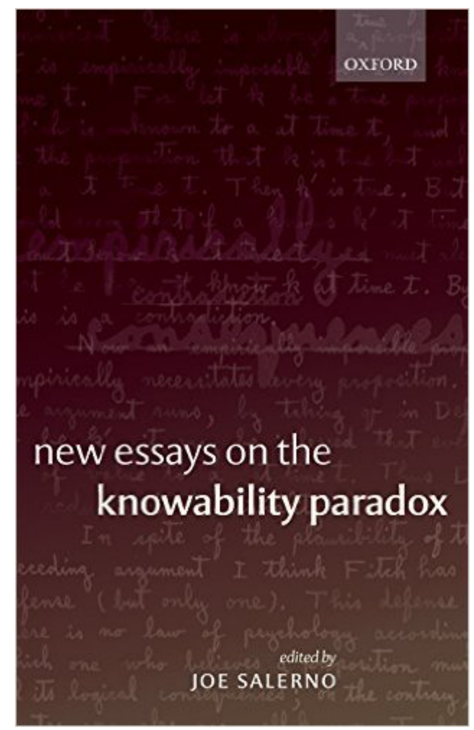
  - Both Kvanvig and Williamson say that if you start with this, you get a parallel deduction that confirms the statement that there's an absolutely unknowable truth.

Further Reading …







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{:name        "Knowability paradox"
 :description " exists p  ~Diamond exists x Kx (Tp & ~ exist y Ky Tp)"

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 :goal (exists [?P] (not (pos (exists [?x] (Knows! ?x (and ?P (not (exists [?y] (Knows! ?y ?P))))))))))}


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Sandbox

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/Library/Java/JavaVirtualMachines/jdk1.8.0_112.jdk/Contents/Home/bin/java ...
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$$\exists \phi \neg \Diamond \exists a \mathbf{K}[a, T(\phi) \wedge \neg \exists a' \mathbf{K}(a', T(\phi))]$$

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slutten