Rationality & Paradox, Part I:
The Liar;
The Barber;
(The Knowability Paradox)

Selmer Bringsjord
Are Humans Rational?
10/23/17
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Types of Paradoxes

- Deductive Paradoxes — corresponding to Area 1, going all the way back to our syllabus
- Inductive Paradoxes — corresponding to Area 2
Types of Paradoxes

Today; Part I

• Deductive Paradoxes — corresponding to Area 1, going all the way back to our syllabus

• Inductive Paradoxes — corresponding to Area 2
The Liar (Paradox) …
First, …

remember/know:
First, ... remember/know:

\[ p \text{ iff } \neg p \text{ (for any statement } p) \]
First, …

remember/know:

\[ p \text{ iff } \neg p \] (for any statement \( p \))

is a contradiction: it’s equivalent to \( p \land \neg p \).
First, …

remember/know:

\( p \) iff \( \neg p \) (for any statement \( p \))

is a contradiction: it’s equivalent to \( p \) & \( \neg p \).

(Why?)
The (Economical) Liar
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L: This sentence is false.
The (Economical) Liar

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If $T(L)$ then $\neg T(L)$
The (Economical) Liar

\[ L: \text{This sentence is false.} \]

\[ \text{If } T(L) \text{ then } \neg T(L) \]

\[ \text{If } \neg T(L) \text{ then } T(L) \]
The (Economical) Liar

L: This sentence is false.

If T(L) then ¬T(L)

If ¬T(L) then T(L)

T(L) iff (i.e., if & only if) ¬T(L)
The (Economical) Liar

L: This sentence is false.

If $T(L)$ then $\neg T(L)$

If $\neg T(L)$ then $T(L)$

$T(L)$ iff (i.e., if & only if) $\neg T(L)$

Contradiction!
The (Verbose) Liar — With a Twist
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Theorem: $2+2 = 5$. 
The (Verbose) Liar — With a Twist

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Proof: Set:
The (Verbose) Liar — With a Twist

Theorem: $2+2 = 5$.

Proof: Set:

$L$: This sentence is false.
The (Verbose) Liar — With a Twist

Theorem: 2+2 = 5.

Proof: Set:

L: This sentence is false.

L is either true or false. Suppose that it’s true. Then since what it says is that it’s false, it is false; i.e., L is false, on this supposition. So we’ve proved that if L is true, L is false. Now suppose instead that L is false. Then since it says that it’s false, it’s true; i.e., L is true, on our current supposition. We have thus proved that if L is false, L is true. Combining the conditionals we’ve proved yields this: L is true if and only if L is false, which is a contradiction. (P if and only if ¬P is logically equivalent to P and ¬P.) By the rule of inference explosion, it follows that 2+2 = 5. QED
Outlawing Self-Referential Sentences Isn’t the Answer!
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• For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
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• For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.

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- For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
  - This sentence is a sentence.
  - This sentence contains the letter ‘r’.
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- For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
  - This sentence is a sentence.
  - This sentence contains the letter ‘r’.
  - This sentence has more than three letters in it.
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• For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.

• This sentence is a sentence.

• This sentence contains the letter ‘r’.

• This sentence has more than three letters in it.

• This sentence ends with a period, starts with a capital ’T’, and has more than two words.
Outlawing Self-Referential Sentences Isn’t the Answer!

- For the following sentences, e.g., are perfectly innocuous, and obviously true or false (only) as the case may be, without complication.
  - This sentence is a sentence.
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  - …
Outlawing Self-Referential Sentences Isn’t the Answer!
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Box 1
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1

Box 2
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1

The sentence in Box 2 is true.

Box 2

The sentence in Box 1 is false.
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1
The sentence in Box 2 is true.

Box 2
Neither sentence is self-referential.

The sentence in Box 1 is false.
Outlawing Self-Referential Sentences Isn’t the Answer!

Box 1

The sentence in Box 2 is true.

Neither sentence is self-referential.

Box 2

The sentence in Box 1 is false.
Suppose that the sentence in Box 1 is true. Then the sentence in Box 2 is true (because the sentence in Box 1 says that that sentence is true). But then the sentence in Box 1 is false (because the sentence in Box 2 says that that sentence is false). So, if the sentence in Box 1 is true, it’s false. On the other hand, if the sentence in Box 1 is false, the sentence in Box 2 is false. But then the sentence in Box 1 is true; so we’ve shown that if the sentence in Box 1 is false, it’s true. We thus have again a contradiction: The sentence in Box 1 is true if and only if it’s not true!
Further Reading ...
The Barber Paradox …
There was once a small town in Norway in which there was a barber who shaved all and only the men residing in the town who didn’t shave themselves.
There was once a small town in Norway in which there was a barber who shaved all and only the men residing in the town who didn’t shave themselves.

There was once a small town in Norway in which there resided a barber who shaved all and only the men residing in the town who didn’t shave themselves.
There was once a small town in Norway in which there was a barber who shaved all and only the men residing in the town who didn’t shave themselves.

There was once a small town in Norway in which there resided a barber who shaved all and only the men residing in the town who didn’t shave themselves.

There was once a small town in Norway in which there resided a male barber who shaved all and only the men residing in the town who didn’t shave themselves.
Such a situation is impossible!
Such a situation is impossible!

Proof: Let’s assume for the sake of argument that such a situation can be. Without loss of generality, let the town be Lyngdal and the male Lyngdalian barber be Olaf. Either Olaf shaves himself or he doesn’t. But either case leads straight to a contradiction. Therefore the situation is in fact impossible. Here we go …

Suppose Olaf shaves himself. Then it follows that he doesn’t shave himself. Suppose on the other hand that Olaf doesn’t shave himself. Then is follows that he does shave himself. Hence, Olaf shaves himself if and only if he doesn’t shave himself, which is a contradiction. QED
• Argument: (1) Harrison admires only great actors who don’t admire themselves. (2) Harrison admires all great actors who don’t admire themselves. Therefore, (3) Harrison isn’t a great actor.

• Purported Proof to Certify the Argument: For indirect proof, suppose that H is a great actor. Either H admires himself, or he doesn’t. We show that either case leads to a contradiction, and hence that our starting supposition is false, i.e. H isn’t a great actor. First, suppose that H does admire himself. From this and (1) we deduce that H doesn’t admire himself — contradiction. For the second case, suppose that H doesn’t admire himself. From this and (2) it follows that Harrison does admire himself — contradiction. Either alternative needs to the desired contradiction. QED

• Is the purported proof in fact a valid proof?
Show that it would be impossible to construct a reference book that lists all and only those reference books that do not list themselves.
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Dr Who and ...
The Problem of the Osirians ...
THE PROBLEM OF THE OSIRIANS
ONE QUESTION ALLOWED!

Impisoned in tube!

B₁ B₂

Sarah

The Twin Guardians of Horus

G₁ G₂

"Contra-programmed;"

one guard always lies; one always tells the truth.

PR WHO

one button, when pushed: instant life / freedom.

one button, when pushed: instant death!
The Problem of the Osirians

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BL BR

The Twin Guardians of Horus

G1 G2

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Pr Who

"If, G2, I were to ask G1 which button is the one for life / freedom, which button would he point to?"
The Problem of the Osirians
One question allowed!

Imprisoned in tube!

B₂, BR

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The Twin Guardians of Horus

G₁

G₂

Contra-programmed:
One guard always lies; one always tells the truth.

One button, when pushed: instant life/freedom.
One button, when pushed: instant death!

If G₂, I were to ask G₁ which button is the one for life/freedom, which button would he point to?
Dr Who’s (Background)
Liar-Leveraging Proof

Proposition: \( B_R \) leads to freedom and life!

Proof: \( G_2 \), who has uttered “\( B_L \),” is either a liar or a truth-teller. Suppose, first, the former case. Then \( G_1 \) would in fact say “\( B_R \)” and would be telling the truth in so saying; hence \( B_R \) on this first supposition is the way to go. What about the latter case? In this case, \( G_2 \) is telling the truth, and hence \( G_1 \) would in fact say “\( B_L \)” — but would be lying, so in this case \( B_R \) is again the way to go. QED
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Gödel proved his first incompleteness theorem, one of the greatest achievements in the history of mathematics, by using reasoning that parallels the reasoning in *The Liar!!*
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Syllabus:Schedule:December 4!
For Further (Supererogatory) Study: The Knowability Paradox …
Don’t blame me. The true source only recently discovered in 2009: Alonzo Church, Turing’s PhD advisor in the States, and the inventor of the lambda-calculus.
The Knowability Paradox (informal)
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Suppose that we know that there is at least one unknown truth; let’s call it p*; this is an entirely arbitrary name. (While this is a supposition, it would be a very plausible assertion. After all, we’re finite, non-omniscient beings. And don’t we have examples?)
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K(p^* \& \neg Kp^*) \quad (1)
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Obviously, knowledge distributes over conjunctions. After all, if someone at some time knows that snow is white and that Tromsø is in Norway, it logically follows that someone at some time knows that snow is white, and someone at some time knows that Tromsø is in Norway. Hence:
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$$Kp^* \& K\neg Kp^* \quad (2)$$
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Next, if someone at some time knows a proposition $s$, then $s$ holds. We can apply this to the right conjunct in (2) to obtain:
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Kp^* \& \neg Kp^* \quad (3)
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Obviously, knowledge distributes over conjunctions. After all, if someone at some time knows that snow is white and that Tromsø is in Norway, it logically follows that someone at some time knows that snow is white, and someone at some time knows that Tromsø is in Norway. Hence:

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But (3) is a contradiction. Hence by *reductio ad absurdum*, we reject the original supposition that led to the contradiction. I.e., we deduce:

$$\neg K(p^* \& \neg Kp^*) \quad (4)$$
The Knowability Paradox (informal)

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$$K(p^* \& \neg Kp^*)$$  \hspace{1cm} (1)

Obviously, knowledge distributes over conjunctions. After all, if someone at some time knows that snow is white and that Tromsø is in Norway, it logically follows that someone at some time knows that snow is white, and someone at some time knows that Tromsø is in Norway. Hence:

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We’re not done yet, not by a long shot! For (4) is an outright theorem, and hence holds necessarily; this yields:
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Next, if someone at some time knows a proposition \( s \), then \( s \) holds. We can apply this to the right conjunct in (2) to obtain:

\[
\neg K(p^* & \neg Kp^*) \quad (4)
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But (3) is a contradiction. Hence by reductio ad absurdum, we reject the original supposition that led to the contradiction. I.e., we deduce:

\[
\neg K(p^* & \neg Kp^*) \quad (5)
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We’re not done yet, not by a long shot! For (4) is an outright theorem, and hence holds necessarily; this yields:

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\text{Necessarily: } \neg K(p^* & \neg Kp^*) \quad (5)
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And in a final move, we know that from ‘necessarily it’s not the case that’ it follows that ‘it’s not possible that it’s the case that’, so we can deduce from (5):
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And in a final move, we know that from ‘necessarily it’s not the case that’ it follows that ‘it’s not possible that it’s the case that’, so we can deduce from (5):

\[
\text{It’s not possible that: } K(p^* \land \neg Kp^*) \tag{6}
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We’re not done yet, not by a long shot! For (4) is an outright theorem, and hence holds necessarily; this yields:

\[
\text{Necessarily: } \neg K(p^* \land \neg Kp^*) \quad (5)
\]

And in a final move, we know that from ‘necessarily it’s not the case that’ it follows that ‘it’s not possible that it’s the case that’, so we can deduce from (5):

\[
\text{It’s not possible that: } K(p^* \land \neg Kp^*) \quad (6)
\]

What (6) says is this: There is an unknownable truth! — for what (6) declares is this: “It’s impossible that someone at some time knows that the conjunction of \( p^* \land \neg Kp^* \) holds.”
The Knowability Paradox (informal)

Suppose that we know that there is at least one unknown truth; let’s call it p*; this is an entirely arbitrary name. (While this is a supposition, it would be a very plausible assertion. After all, we’re finite, non-omniscient beings. And don’t we have examples?)

Letting $K$ represent ‘someone at some time knows,’ our supposition can be conveniently represented as the statement:

$$K(p^* \& \neg Kp^*) \quad (1)$$

Obviously, knowledge distributes over conjunctions. After all, if someone at some time knows that snow is white and that Tromsø is in Norway, it logically follows that someone at some time knows that snow is white, and someone at some time knows that Tromsø is in Norway. Hence:

$$Kp^* \& K\neg Kp^* \quad (2)$$

Next, if someone at some time knows a proposition s, then s holds. We can apply this to the right conjunct in (2) to obtain:

$$Kp^* \& \neg Kp^* \quad (3)$$

But (3) is a contradiction. Hence by reductio ad absurdum, we reject the original supposition that led to the contradiction. I.e., we deduce:

$$\neg K(p^* \& \neg Kp^*) \quad (4)$$

We’re not done yet, not by a long shot! For (4) is an outright theorem, and hence holds necessarily; this yields:

Necessarily: $$\neg K(p^* \& \neg Kp^*) \quad (5)$$

And in a final move, we know that from ‘necessarily it’s not the case that’ it follows that ‘it’s not possible that it’s the case that’, so we can deduce from (5):

It’s not possible that: $$K(p^* \& \neg Kp^*) \quad (6)$$

What (6) says is this: There is an unknowable truth! — for what (6) declares is this: “It’s impossible that someone at some time knows that the conjunction of $p^*$ & $\neg Kp^*$ holds.” !
The Knowability Paradox (informal)

Suppose that we know that there is at least one unknown truth; let’s call it p*; this is an entirely arbitrary name. (While this is a supposition, it would be a very plausible assertion. After all, we’re finite, non-omniscient beings. And don’t we have examples?) Letting \( K \) represent ‘someone at some time knows,’ our supposition can be conveniently represented as the statement:

\[
K(p* \& \neg Kp*) \quad (1)
\]

Obviously, knowledge distributes over conjunctions. After all, if someone at some time knows that snow is white and that Tromsø is in Norway, it logically follows that someone at some time knows that snow is white, and someone at some time knows that Tromsø is in Norway. Hence:

\[
Kp* \& \neg Kp* \quad (2)
\]

But (3) is a contradiction. Hence by *reductio ad absurdum*, we reject the original supposition that led to the contradiction. I.e., we deduce:

\[
\neg K(p* \& \neg Kp*) \quad (4)
\]

We’re not done yet, not by a long shot! For (4) is an outright theorem, and hence holds necessarily; this yields:

Necessarily: \( \neg K(p* \& \neg Kp*) \quad (5) \)

And in a final move, we know that from ‘necessarily it’s not the case that’ it follows that ‘it’s not possible that it’s the case that’, so we can deduce from (5):

It’s not possible that: \( K(p* \& \neg Kp*) \quad (6) \)

What (6) says is this: There is an *unknowable* truth! — for what (6) declares is this: “It’s impossible that someone at some time knows that the conjunction of p* & \( \neg Kp* \) holds.”!
The Knowability Paradox (informal)

Suppose that we know that there is at least one unknown truth; let's call it p*; this is an entirely arbitrary name. (While this is a supposition, it would be a very plausible assertion. After all, we’re finite, non-omniscient beings. And don’t we have examples?)

Letting \( K \) represent 'someone at some time knows,' our supposition can be conveniently represented as the statement:

\[
K(p* \& \neg Kp*) \quad (1)
\]

Obviously, knowledge distributes over conjunctions. After all, if someone at some time knows that snow is white and that Tromsø is in Norway, it logically follows that someone at some time knows that snow is white, and someone at some time knows that Tromsø is in Norway. Hence:

\[
Kp* \& K\neg Kp* \quad (2)
\]

But (3) is a contradiction. Hence by reductio ad absurdum, we reject the original supposition that led to the contradiction. I.e., we deduce:

\[
K(p* \& \neg Kp*) \quad (3)
\]

Next, if someone at some time knows a proposition \( s \), then \( s \) holds. We can apply this to the right conjunct in (2) to obtain:

\[
\neg K(p* \& \neg Kp*) \quad (4)
\]

We’re not done yet, not by a long shot. For (4) is an outright theorem, and hence holds necessarily; this yields:

\[
\text{Necessarily: } \neg K(p* \& \neg Kp*) \quad (5)
\]

And in a final move, we know that from ‘necessarily it’s not the case that’ it follows that ‘it’s not possible that it’s the case that,’ so we can deduce that — and this is (6) — there’s a truth that is absolutely unknowable.

\[
\text{It's not possible that: } K(p* \& \neg Kp*) \quad (6)
\]

What (6) says is this: There is an unknowable truth! — for what (6) declares is this: “It’s impossible that someone at some time knows that the conjunction of \( p* \& \neg Kp* \) holds.”!
Invalid?; Solutions
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- Apparent unanimity that no escape by finding a technical error that invalidates the reasoning.
Invalid?; Solutions

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- E.g., Kvanvig says this, & Williamson concurs. (See further reading on next slide.)
Invalid?; Solutions

- Apparent unanimity that no escape by finding a technical error that invalidates the reasoning.
  
  - E.g., Kvanvig says this, & Williamson concurs. (See further reading on next slide.)

- Specifically, John Andrews, Lauren Talbot, Hannah F. were concerned that the initial statement isn’t something like: “There exists some proposition $p$ which is such that, $p$ holds, and no one knows that $p$.”
Invalid?; Solutions

• Apparent unanimity that no escape by finding a technical error that invalidates the reasoning.

• E.g., Kvanvig says this, & Williamson concurs. (See further reading on next slide.)

• Specifically, John Andrews, Lauren Talbot, Hannah F. were concerned that the initial statement isn’t something like: “There exists some proposition $p$ which is such that, $p$ holds, and no one knows that $p$.”

• Both Kvanvig and Williamson say that if you start with this, you get a parallel deduction that confirms the statement that there’s an absolutely unknowable truth.
Further Reading ...
The Knowability Paradox in the Slate System

Selmer Bringsjord

No_one_knows that (q*):_p*_and_no_one_knows_p*. ¬□(p* ∧ ¬□p*)

Proposition q* is unknowable. □¬(p* ∧ ¬□p*)
Suppose that there’s an unknown truth; let’s call it ‘p*’.

No_one_knows that (q*):_p*_and_no_one_knows_p*. ¬(p* ∧ ¬p*)

Proposition q* is unknowable. □¬(p* ∧ ¬p*)
Suppose that there’s an unknown truth; let’s call it ‘p*’.

(The box here stands for ‘Knows’; more specifically, for ‘someone at some time knows that’.)

1. □(p* ∧ ¬□p*)
   \{1\} Assume \(\checkmark\) \(\square\)

2. □p* ∧ □¬□p*
   \{1\} \(\square\)

3. □¬□p*
   \{1\} \(\square\)

4. ¬□p*
   \{1\} \(\square\)

5. □p*
   \{1\} \(\square\)

\(\wedge\) elim \(\checkmark\)

No_one_knows that (q*):_p*_and_no_one_knows_p*. ¬□(p* ∧ ¬□p*)

\(\square\) intro \(\checkmark\)

Proposition q* is unknowable. □¬□(p* ∧ ¬□p*)

\(\square\) intro \(\checkmark\)

Selmer Bringsjord
Suppose that there’s an unknown truth; let’s call it ‘p*’.

(The box here stands for ‘Knows’; more specifically, for ‘someone at some time knows that’.)

1. □(p* ∧ ¬□p*)
   {1} Assume ✓ \(\square\)

2. □p* ∧ □¬□p*
   {1} \(\square\)

3. □¬□p*
   {1} \(\square\)

4. ¬□p*
   {1} \(\square\)

5. □p*
   {1} \(\square\)

¬ intro ✓

No_one_knows that (q*): _p*_ and no_one_knows_p*. ¬□(p* ∧ ¬□p*)

□ intro ✓

Proposition q* is unknowable. □¬□(p* ∧ ¬□p*)

Selmer Bringsjord
Suppose that there’s an unknown truth; let’s call it ‘p*’. (The box here stands for ‘Knows’; more specifically, for ‘someone at some time knows that’.)

1. □(p* ∧ ¬ □ p*)
   (1) Assume ✓ ∞ □

2. □ p* ∧ □ ¬ □ p*
   (1) ∞ □

   ∧ elim ✓

   3. □ ¬ □ p*
      (1) ∞ □

   ∧ elim ✓

   5. □ p*
      (1) ∞ □

   □ elim ✓

   4. ¬ □ p*
      (1) ∞ □

¬ intro ✓

No_one_knows that (q*): _p*_ and_no_one_knows_p*. ¬ □(p* ∧ ¬ □ p*)

□ intro ✓

Proposition q* is unknowable. □ ¬ □(p* ∧ ¬ □ p*)

Selmer Bringsjord

The Knowability Paradox in the Slate System
Suppose that there's an unknown truth; let's call it 'p*'.

Distribute the knowledge operator into the conjunction.

Extract the right conjunct.

Extract the left conjunct.

No one knows that (q*): p* and no one knows p*. ∼ □(p* ∧ ∼ □p*).

Proposition q* is unknowable. □ ∼ □(p* ∧ ∼ □p*)
Suppose that there’s an unknown truth; let’s call it ‘p*’.

1. \(\square(p^* \land \neg \square p^*)\)
   \{1\} Assume \(\checkmark\) \(\neg \square\)

2. \(\square p^* \land \neg \square p^*\)
   \{1\} \(\checkmark\) \(\neg \square\)

   \(\land\) elim \(\checkmark\)

3. \(\square \neg p^*\)
   \{1\} \(\checkmark\) \(\neg \square\)

   \(\square\) elim \(\checkmark\)

4. \(\neg \square p^*\)
   \{1\} \(\checkmark\) \(\neg \square\)

   \(\square\) intro \(\checkmark\)

5. \(\square p^*\)
   \{1\} \(\checkmark\) \(\neg \square\)

   \(\land\) elim \(\checkmark\)

\(\neg\) intro \(\checkmark\)

\(\square\) intro \(\checkmark\)

No_one_knows that (q*): \_p*_and_no_one_knows_p*. \(\neg \square(p^* \land \neg \square p^*)\)

Proposition q* is unknowable. \(\square \neg (p^* \land \neg \square p^*)\)

(The box here stands for ‘Knows’; more specifically, for ‘someone at some time knows that’.)
Suppose that there's an unknown truth; let's call it 'p*'.

1. \(\square(p^* \land \neg \square p^*)\)  
   \(\{1\}\) Assume \(\forall \square\)

2. \(\square p^* \land \square \neg \square p^*\)  
   \(\{1\}\) \(\forall \square\)

   \(\wedge\) elim \(\checkmark\)

3. \(\square \neg \square p^*\)  
   \(\{1\}\) \(\forall \square\)

   \(\neg\) elim \(\checkmark\)

4. \(\neg \square p^*\)  
   \(\{1\}\) \(\forall \square\)

   \(\neg\) intro \(\checkmark\)

   No_one_knows that (q*): \_p*_and_no_one_knows_p*. \(\neg \square(p^* \land \neg \square p^*)\)  
   \(\forall \square\)

   \(\square\) intro \(\checkmark\)

   Proposition q* is unknowable. \(\square \neg \square(p^* \land \neg \square p^*)\)  
   \(\forall \square\)

(The box here stands for ‘Knows’; more specifically, for ‘someone at some time knows that’.)

Distribute the knowledge operator into the conjunction.

Extract the right conjunct.

Extract the left conjunct.

By indirect proof (reductio ad absurdum) our original supposition is negated.

Selmer Bringsjord
Suppose that there’s an unknown truth; let’s call it ‘p*'.

1. □(p* ∧ ¬□p*) 
   {1} Assume ∨ □

(The box here stands for ‘Knows’; more specifically, for ‘someone at some time knows that’.)

2. □p* ∧ □¬□p* 
   {1} ∨ □

Distribute the knowledge operator into the conjunction.

3. □¬□p* 
   {1} □

Extract the right conjunct.

4. ¬□p* 
   {1} □

Whatever is known is the case.

5. □p* 
   {1} □

Extract the left conjunct.

Distribute the knowledge operator into the conjunction.

No_one knows that (q*): _p*_and_no_one_knows_p*. ¬□(p* ∧ ¬□p*)

By indirect proof (reductio ad absurdum) our original supposition is negated.

But since we’ just proved a theorem (there are no remaining suppositions), we can deduce that some proposition (let’s call it q*) is unknowable [because we add a box (this time to represent ‘necessarily’)].
Knowability Paradox

\[ \exists x \Diamond (Kx \land \exists y (T_p \land \neg T_y)) \]

Assumptions:
- \( T_p \land \neg T_y \land \neg \exists x (Kx \land \exists y (T_y \land \neg T_y)) \)
- Pos (\( \exists x (Kx \land \exists y (T_y \land \neg T_y)) \))
Knowability paradox

\[ \exists x \, (Kx \land (\exists y \, (Kp \land \neg Kp))) \]

assumptions ()

goal (exists ?P (not (pos (exists (?x (Knows ! ?x) and ?P (not (exists (?y (Knows ! ?y)))))))))
\[\exists \phi \to \Box \exists a K[a, T(\phi) \land \neg \exists a' K(a', T(\phi))]\]
slutten