Steeple #3:
Gödel’s “Silver Blaze” Theorem

Selmer Bringsjord
Are Humans Rational?
Dec 6 2018
RPI
Troy NY USA
• Introduction (“The Wager”)
• Brief Preliminaries (e.g. the propositional calculus)
• The Completeness Theorem
• The First Incompleteness Theorem
• The Second Incompleteness Theorem
• The Speedup Theorem
• The Continuum-Hypothesis Theorem
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ZFC  (The Foundation of Mathematics)
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AC: Given a set $A$ of nonempty pairwise disjoint sets, there exists a set which contains exactly one element of each set in $A$. 
Arithmetic is Part of All Things Sci/Eng/Tech!

and courtesy of Gödel: We can’t even prove all truths of arithmetic!

Each circle is a larger part of the formal sciences.
Actually, the true kernel is set theory!

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Cantor’s Theorem
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Cantor (1878):
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The power set of the natural numbers is larger than the natural numbers!

How do we know this??!??!
Continuum Hypothesis
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$\text{CH} : \forall S[(S \subset \mathbb{R} \land \neg \text{Fin}(S)) \rightarrow (S \sim \mathbb{N} \lor S \sim \mathbb{R})]$
Continuum Hypothesis

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Every infinite subset of the reals is either the same size as the natural numbers or the same size as the reals.
Generalized Continuum Hypothesis
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For every infinite set $S$, $\mathcal{P}(S) > S$. 
Generalized Continuum Hypothesis

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Generalized Continuum Hypothesis (GCH):

There’s no set (size-wise) between $S$ and $P(S)$. 
“Shorthand” History, Moral

Hilbert’s #1 (1900): “very plausible theorem”: CH
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ADR-based approach to enabling an AI to discover and prove Gödel’s First Incompleteness Theorem:
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ADR-based approach to enabling an AI to discover and prove Gödel’s First Incompleteness Theorem:

Won’t work on this theorem of Gödel’s!
Fleshing Out a Bit …
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Facts F ...
Scenario G
(in honor of Inspector Gregory)

“And yet,” said I, “even now I fail to understand what the theory of the police can be.”

“I am afraid that whatever theory we state has very grave objections to it,” returned my companion. “The police imagine, I take it, that this Fitzroy Simpson, having drugged the lad, and having in some way obtained a duplicate key, opened the stable door and took out the horse, with the intention, apparently, of kidnapping him altogether. His bridle is missing, so that Simpson must have put this on. Then, having left the door open behind him, he was leading the horse away over the moor, when he was either met or overtaken by the trainer. A row naturally ensued. Simpson beat out the trainer’s brains with his heavy stick without receiving any injury from the small knife which Straker used in self-defense, and then the thief either led the horse on to some secret hiding-place, or else it may have bolted during the struggle, and be now wandering out on the moors. That is the case as it appears to the police, and improbable as it is, all other explanations are more improbable still. However, I shall very quickly test the matter when I am once upon the spot, and until then I cannot really see how we can get much further than our present position.”
Gödel’s Proof

**Proof-Sketch:** GCH is equivalent to the statement that there are at most $c^*$ subsets of $c$, where ‘$c$’ is any cardinal number, and $c^*$ is the cardinal after $c$ (i.e., $|P(c)| \leq c^*$). Gödel showed that there are at most $c^*$ constructible subsets of $c$; hence if all sets are constructible, GCH follows. Gödel created a scenario in which all sets are constructible, and showed that this scenario is consistent with ZFC (assuming, as we are, that ZFC is consistent). Therefore GCH can never be disproved from ZFC! \( \textbf{QED} \)
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(Of course, in “Silver Blaze,” Sherlock did proceed to show that his scenario was true. The analogue to that doesn’t hold in the Gödel story.)
Main Claim

Humans, at least neurobiologically normal ones, are fundamentally rational, where rationality is constituted by certain logico-mathematically based reasoning and decision-making in response to real-world stimuli, including stimuli given in the form of focused tests; but mere animals are not fundamentally rational, since, contra Darwin, their minds are fundamentally qualitatively inferior to the human mind. As to whether computing machines/robots are fundamentally rational, the answer is also “No.” For starters, if $x$ can’t read, write, and create, $x$ can’t be rational; neither computing machines/robots nor non-human animals can read nor write nor create; ergo, they aren’t fundamentally rational for this reason alone. But news for non-human animals and computing machines/robots gets much worse, for they have not the slightest chance when they are measured against $H$. 
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And Supporting Main Claim …

$H$ Humans have the ability to gain knowledge by reasoning (e.g., deductively) quantificationally and recursively over abstract concepts, including abstract concepts of a highly expressive, including infinitary, nature, expressed in arbitrarily complex natural language.
Re Papers

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