Steeple #3:
Gödel's "Silver Blaze" Theorem

Selmer Bringsjord

Are Humans Rational?

Dec 8 2019

RPI
Troy NY USA

Rensselaer AI and Reasoning Lab

Gödels' Great Theorems (OUP)

by Selmer Bringsjord

- Introduction ("The Wager")
- Brief Preliminaries (e.g. the propositional calculus)
- The Completeness Theorem
- The First Incompleteness Theorem
- The Second Incompleteness Theorem
- The Speedup Theorem
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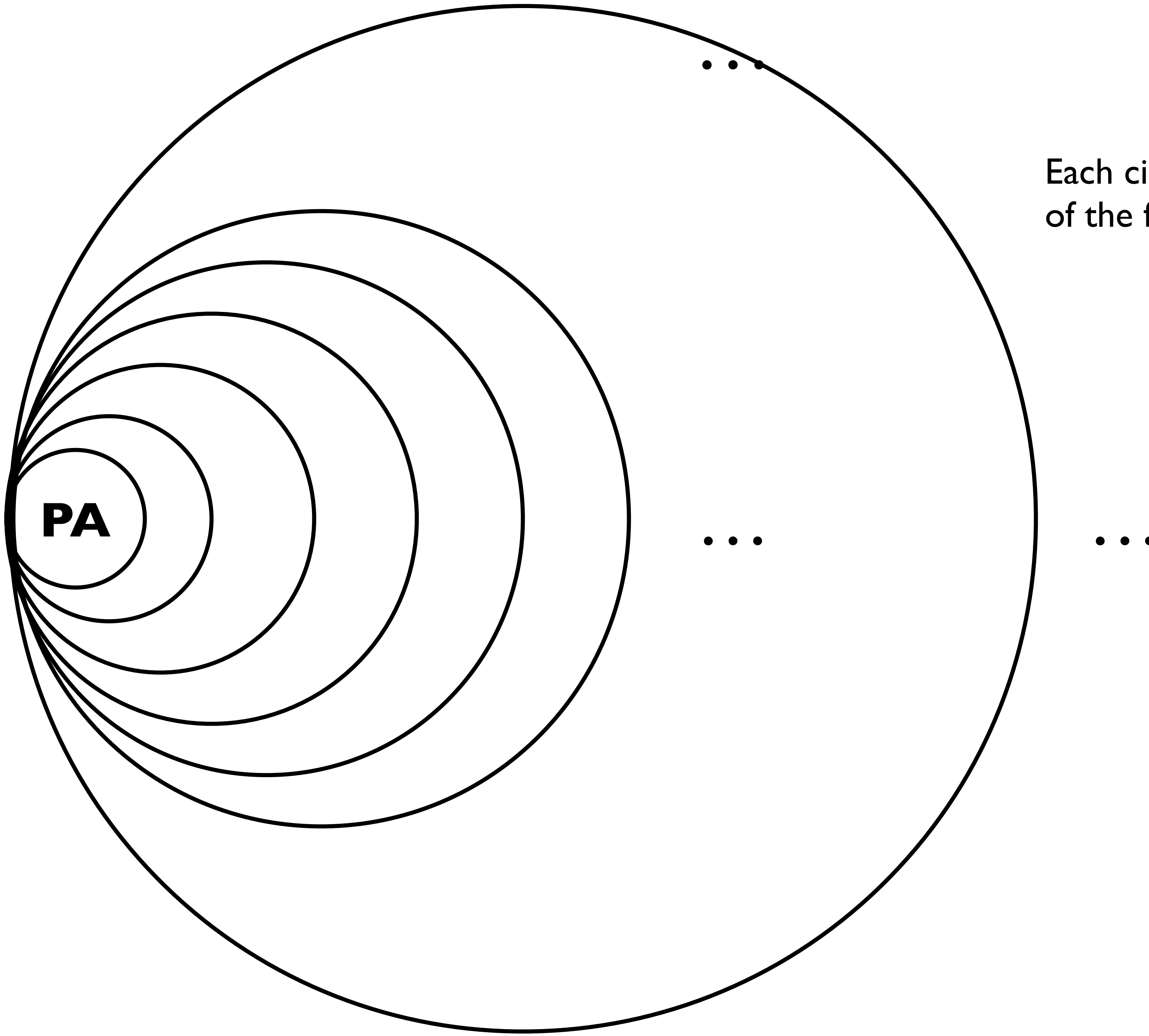
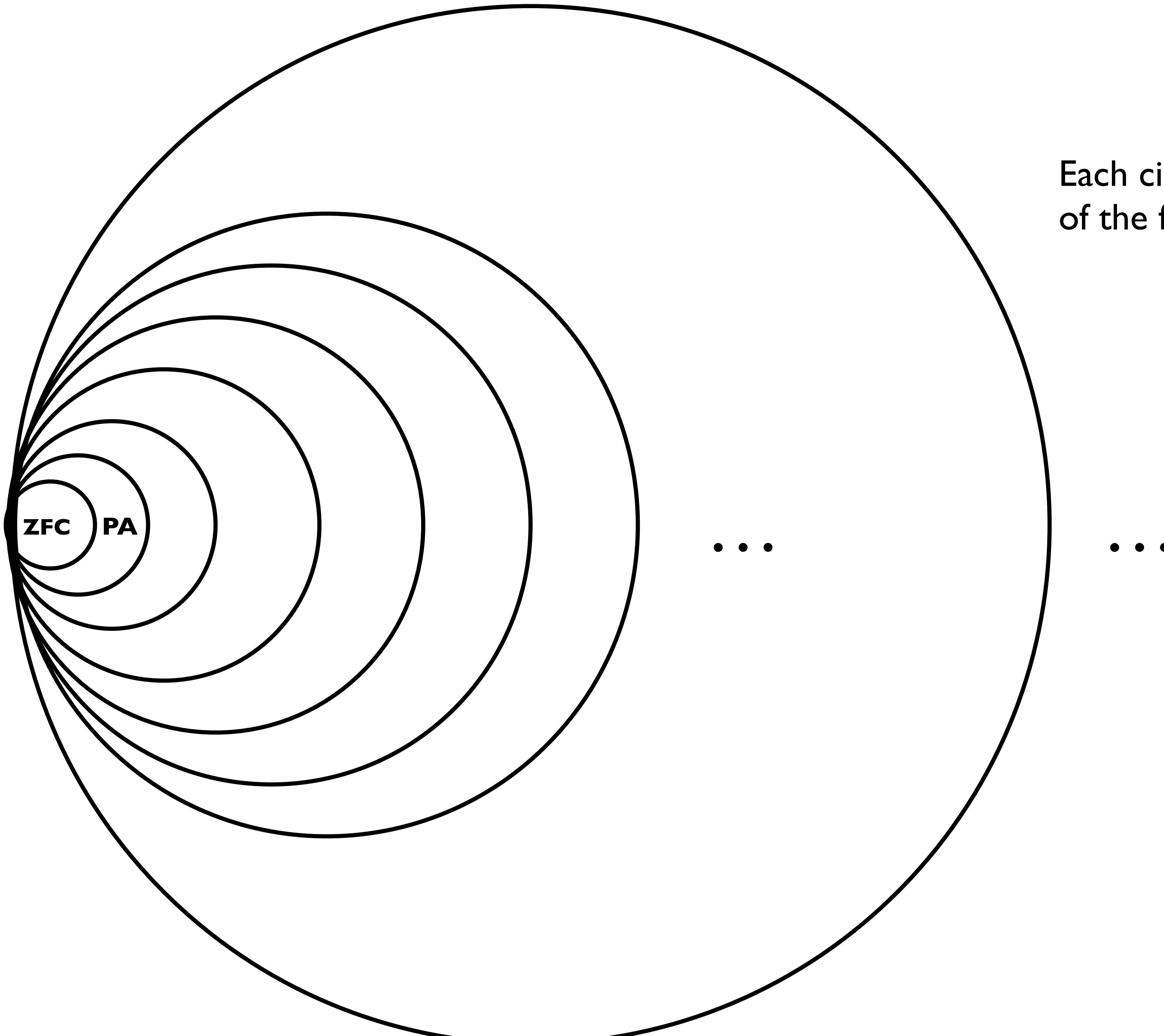
Arithmetic is Part of All Things Sci/Eng/Tech!

and courtesy of Gödel: We can't even prove all truths of arithmetic!



Each circle is a larger part of the formal sciences.

Actually, the true kernel is set theory!



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How do we know this???!???

Continuum Hypothesis

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$$\mathrm{CH}: \forall S[(S \subset \mathbb{R} \wedge \neg \mathbf{Fin}(S)) \to (S \sim \mathbb{N} \vee S \sim \mathbb{R})]$$

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$$\mathrm{CH} : \forall S[(S \subset \mathbb{R} \land \neg \mathbf{Fin}(S)) \to (S \sim \mathbb{N} \lor S \sim \mathbb{R})]$$

Every infinite subset of the reals is either the same size as the natural numbers or the same size as the reals.

Generalized Continuum Hypothesis

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For every infinite set $S, \mathcal{P}(S) > S$.

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Generalized Continuum Hypothesis (GCH):

There's no set (size-wise) between *S* and *P*(*S*).

“Shorthand” History, Moral

Hilbert’s #1 (1900): “very plausible theorem”: **CH**

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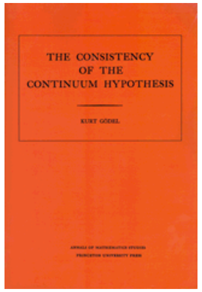

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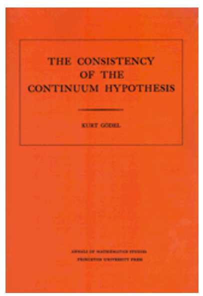

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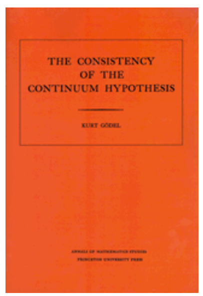

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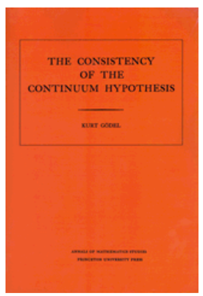

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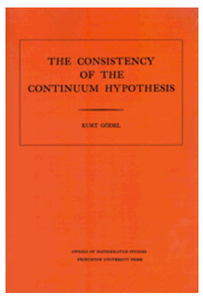

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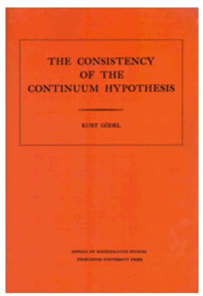

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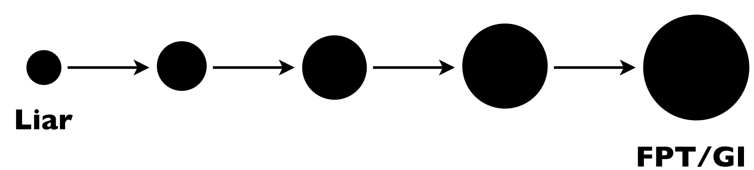

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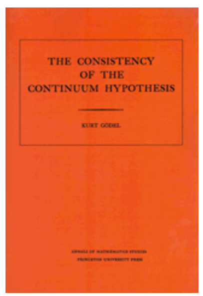

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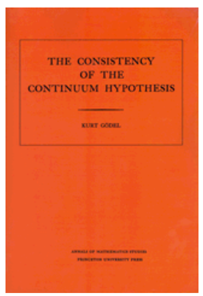

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Won’t work on this theorem of Gödel’s!

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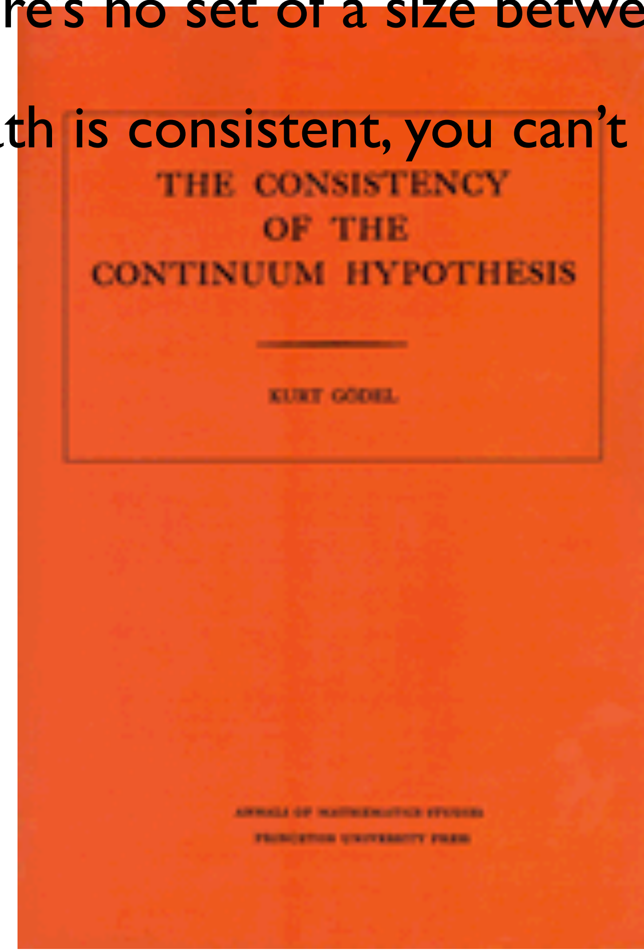

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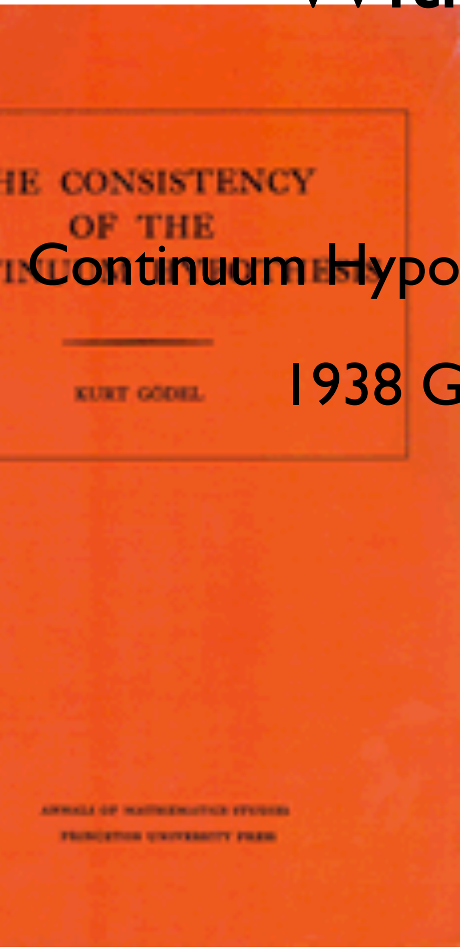

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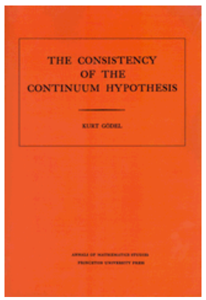

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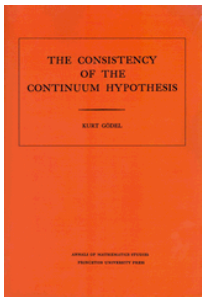

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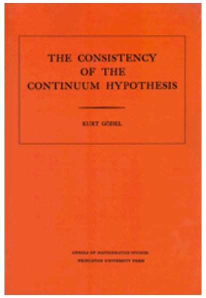

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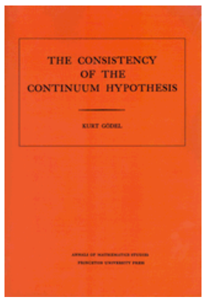

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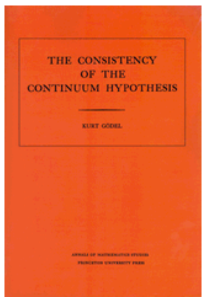

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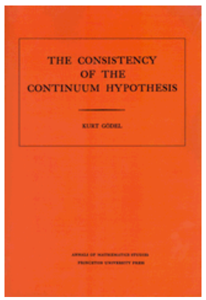

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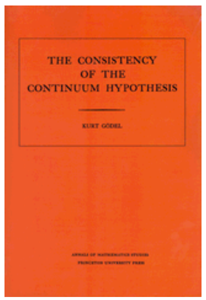

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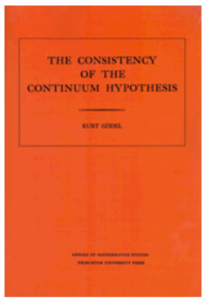

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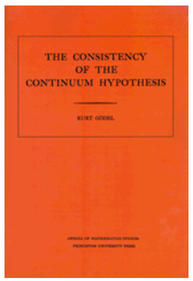

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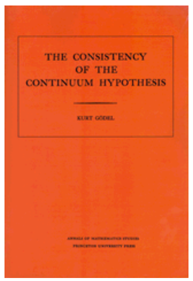

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Facts F …

Scenario G
(in honor of Inspector Gregory)

"And yet," said I, "even now I fail to understand what the theory of the police can be."

"I am afraid that whatever theory we state has very grave objections to it," returned my companion. "The police imagine, I take it, that this Fitzroy Simpson, having drugged the lad, and having in some way obtained a duplicate key, opened the stable door and took out the horse, with the intention, apparently, of kidnapping him altogether. His bridle is missing, so that Simpson must have put this on. Then, having left the door open behind him, he was leading the horse away over the moor, when he was either met or overtaken by the trainer. A row naturally ensued. Simpson beat out the trainer's brains with his heavy stick without receiving any injury from the small knife which Straker used in self-defense, and then the thief either led the horse on to some secret hiding-place, or else it may have bolted during the struggle, and be now wandering out on the moors. That is the case as it appears to the police, and improbable as it is, all other explanations are more improbable still. However, I shall very quickly test the matter when I am once upon the spot, and until then I cannot really see how we can get much further than our present position."

Sherlock as Logician

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F = facts of the case
G = Inspector Gregory's scenario
H = Holmes's scenario

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Holmes: "Gregory's claim is G (Simpson is guilty, & other details re. what he did). We have F, the facts of the case, disputed by no one. The question is: F $\vdash$ G? The answer is clearly No, for my scenario H, which entails $\neg G$, is consistent with the facts [Con $(F \cup H)$], and H entails $\neg G$. Here's the proof: Suppose for *reductio* that F $\vdash$ G. Then F $\cup$ H are inconsistent — contradiction! Hence F $\nvdash$ G."

Gödel's "Holmesian" Proof

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Proof-Sketch: GCH is equivalent to the statement that there are at most *c** subsets of *c*, where '*c*' is any cardinal number, and *c** is the cardinal after *c* (i.e., $|P(c)| \leq c^*$). Gödel showed by employing transfinite recursion (!) that there are at most *c** constructible subsets of *c;* hence if all all sets are constructible, GCH follows. Gödel created a scenario in which all sets *are* constructible (viz., C), and showed that this scenario is consistent with ZFC (assuming, as we are, that ZFC is consistent). Therefore, by parallel to Holmes's reasoning, GCH can never be disproved from ZFC! Here''s the reasoning: Suppose for reductio that $\text{ZFC} \vdash \neg\text{GCH}$. We know that $\text{C} \vdash \text{GCH}$. Therefore C and ZFC are inconsistent; but this contradicts the consistency that Gödel proved. **QED**

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(Of course, in "Silver Blaze," Sherlock did proceed to show that his scenario was true. The analogue to that doesn't hold in the Gödel story.)

slutten

Test 3 …

Question 1 (required) ...

$\mathcal{R}$ Humans, at least when neurobiologically normal and sane, are potentially presently (i.e., to abbreviate this double-adverb mouthful, *fundamentally*) rational, where such rationality is operationalized by certain explicit logico-mathematically based reasoning and decision-making successfully provided in response to real-world stimuli, especially such stimuli as are given in the form of focused, academic-style tests. However, mere animals are *not* fundamentally rational, since, *contra* Darwin,[1] their minds — unable as they are to e.g. reason in accord with abstract inference schemata like *modus tollens* — are provably qualitatively inferior to the human mind, as empirically confirmed by their poor performance on relevant focused, academic-style tests. As to whether computing machines/AIs/robots are fundamentally rational, the answer is also "No." For starters, if x can't read, really learn, write, and create, x can't be fundamentally rational; neither computing machines/robots nor non-human animals can read, really learn, write, or create; ergo, they aren't fundamentally rational for this reason alone. But the news for non-human animals and computing machines/robots gets much worse, for they have not the slightest chance when they are measured against the activity called out in $\mathcal{H}$.

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1: No, humans aren't fundamentally rational as asserted here! & here's why!

2: No, nonhuman animals aren't non-rational as asserted here! & here's why!

3: No, AIs aren't non-rational as asserted here! & here's why!

4: No, the capacities called out in *H* are not beyond nonhuman animals and AIs.

Question 2 (optional) …

Indirect Proof

GI: g^* isn't provable from **PA**; nor is the negation of g^*!

Proof: Let's follow The Liar: Suppose that g^* *is* provable from PA; i.e., suppose **PA** |- g^*. Then by (1), with g* substituted for s, we have:

PA |- q^*("g^*") iff **PA** |- g^* (1')

From our supposition and working right to left by *modus ponens* on (1') we deduce:

PA |- q^*("g^*") (3.1)

But from our supposition and the earlier (see previous slide) (2), we can deduce by *modus ponens* that from PA the opposite can be proved! I.e., we have:

PA |- not-q^*("g^*") (3.2)

But (3.1) and (3.2) together means that **PA** is inconsistent, since it generates a contradiction. But we are working under the supposition that **PA** is consistent. Hence by indirect proof g^* is *not* provable from **PA**.

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Show, in a manner that follows the reasoning in The Liar, that supposing that the negation of g^* (i.e., not-g^*) is provable from **PA** leads to a contradiction, and hence can't be provable from **PA**.

Paper Turn-In Logistics …

Email final draft of paper as a pdf (no other formats accepted) attachment to *both* Selmer.Bringsjord@gmail.com and can.mekik@gmail.com using verbatim the SUBJECT "AHR? S19 Paper Final Draft <firstname> <lastname>" by 1159pm Dec 17 2019.

Thank you. Its been an honor to teach this class.