AHR F19: Test 1 Post-Test

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General Remarks

Read questions **very carefully**, if you misunderstand the question (e.g., you miss a '$\neg$') you can't answer it correctly!

Memorize PC symbols, truth tables, and inference rules.

When in doubt, try proof by cases. It can only help. If doesn't work, try for explosion/contradiction.

Be careful with parentheses. E.g., $\neg(p \to q) \neq \neg p \to q$!

Q1: Statement

Given the following:

$$\neg(p \to q)$$

$$\neg\neg\neg p$$

$$(r \vee s) \to \neg q$$

$$r \vee s$$

What can be deduced?

$\neg u$?　　$\neg p$?　　p?　　h?　　All?

Q1: Answer

The answer is all of the above.

Proof.

From premise $\neg(p \to q)$ we can infer $p \wedge \neg q$, since an implication is only ever false when the antecedent is true and the consequent is false. And from $p \wedge \neg q$ we can infer p using the simplication rule of inference. Finally, from premise $\neg\neg\neg p$ we can infer $\neg p$ using the double negation rule, since there is an odd number of negation symbols. From p and $\neg p$ we can get $p \wedge \neg p$ using the conjunction rule. Then, using the explosion rule, we can prove any arbitrary proposition q'. $\square$

Q2: Statement

Can anything be deduced using PC inference rules from **just** p? If yes, give 5 examples.

Q2: Answer

Yes. In fact, an infinite number of new statements can be deduced using applicable inference rules that introduce new, arbitrary propositions. For example:

1. $p \vee q$ (Addition)
2. $q \to p$, since, by definition, the implication $q \to p$ is true if and only if $q \wedge \neg p$ is false. Since p, $q \wedge \neg p$ is false. Therefore, $q \to p$.
3. $\neg p \to q$, since, by definition, the implication $\neg p \to q$ is true if and only if $\neg p \wedge \neg q$ is false. Since p, $\neg p \wedge \neg q$ is false. Therefore, $q \to p$.
4. $p \vee q \vee r$ (Addition)
5. $p \vee \neg p$ (Addition)

Q3: Statement

Jill claims $\Pr(\mathrm{FTA}) = \Pr(2+2=4)$ Tony disagrees.

Who is right? Justify.

Q3: Answer

Jill is right.

Proof.

The FTA is a theorem, consequently FTA is decductively provable, i.e., $\vdash FTA$. Likewise, '$2+2=4$' is a theorem, i.e., $\vdash$ '$2+2=4'$. According to Kolmogorov's axioms (axiom 2, specifically), if we have $\vdash p$, then $P(p)=1$. Therefore, $P(FTA) = P(`2+2=4') = 1$. $\square$

Q4: Statement

Given

$$\neg(C_S \leftrightarrow J_W)$$
$$\neg(\neg H_S \vee K_I)$$
$$C_S \vee C_S$$

What can Sherlock deduce about each person's location?

Justify your answer, prove any new inference rules you use.

Q4: Answer

Here is what Sherlock knows:

- Jill not in Westchester ($\neg J_W$).
- Chris in Seattle (C_S).
- Henry in Seattle (H_S)
- Kate not in Ireland ($\neg K_I$).

Proofs? A lot of work...

Q4: Proof Plan

We know:

$$\neg(C_S \leftrightarrow J_W)$$
$$\neg(\neg H_S \vee K_I)$$
$$C_S \vee C_S$$

But, to write the proofs, we first need to prove some new rules of inference. For this question, we prove three lemmas (helper theorems/proofs) introducing new rules of inference:

$$\{p \vee p\} \vdash_{PC} p$$
$$\{\neg(p \leftrightarrow q)\} \vdash_{PC} p \oplus q$$
$$\neg(p \vee q) \vdash_{PC} \neg p \wedge \neg q$$

Then, we prove each proposition using lemmas and givens.

Q4: Proof of Lemma 1

By definition, $p \vee p$ holds if and only if at least one of the disjuncts hold. We use proof by cases on the disjuncts; there are three cases: only the left-hand disjunct holds, only the right-hand disjunct holds, or both disjuncts hold. If the left-hand disjunct holds, p holds, likewise if the right-hand disjunct holds p holds. If both disjuncts hold, then p holds trivially by the reasoning above. Therefore, in all cases, if $p \vee p$ holds then p holds.

Q4: Proof of Lemma 2

By definition, $p \leftrightarrow q$ is true if p and q are both true or both false. Therefore, $\neg(p \leftrightarrow q)$ is true if only exactly one of p or q is true. Incidentally, by definition, $p \oplus q$ is true iff only exactly one of p or q is true. Therefore, from $\neg(p \leftrightarrow q)$, we can infer $p \oplus q$.

Q4: Proof of Lemma 3

By definition, $p \vee q$ is true if at least one of the disjuncts are true. Therefore, $\neg(p \vee q)$ is true only if both disjuncts are false. That is, $\neg(p \vee q)$ is true if both $\neg p$ and $\neg q$ are true. Thus, using the conjunction rule, from $\neg p$ and $\neg q$ we can conclude $\neg p \wedge \neg q$.

Q4: Proof of Propositions (Part 1)

Chris is in seattle (i.e., C_S).

Proof.
From premise (c), we have $C_S \vee C_S$. Therefore, we have C_S by Lemma 1. $\square$

Jill is not in Westchester (i.e, $\neg J_W$).

Proof.
From premise (a), we know $\neg(C_S \leftrightarrow J_W)$. Using Lemma 2, we can then infer $C_S \oplus J_W$. By Proposition 1, we know C_S. Since, by definition, exactly one disjunct must hold in an exclusive disjunction, from $C_S \oplus J_W$ and C_W, we may conclude $\neg J_W$. $\square$

Q4: Proof of Propositions (Part 2)

Kate is not in Ireland (i.e., $\neg K_I$).

Proof.
From premise (b), we know $\neg(\neg H_S \vee K_I)$. Therefore, using Lemma 3, we can infer $\neg\neg H_S \wedge \neg K_I$. Using the simplification rule, we then get $\neg K_I$. $\square$

Henry is in Seattle (i.e., H_S).

Proof.
Using the same line of reasoning in Proposition 3, we can infer $\neg\neg H_S$. Then, using the double negation rule, we get H_S. $\square$

Q5: Statement

If one true, so are the other two:

- Smart giraffe in zoo iff male horse in zoo
- Smart giraffe in zoo.
- Lazy llama in zoo.

Which, if any, of smart giraffe, male horse, lazy llama more likely to be in zoo?

Q5: Answer & Proof Setup

The male horse is most likely to be in the zoo.
We proceed using proof by cases. Let:

- G stand for 'There is a smart giraffe in the zoo'
- H stand for 'There is a male horse in the zoo'
- L stand for 'There is a lazy llama in the zoo'

Then we can formalize the sub-propositions thus:

1. $G \leftrightarrow H$
2. G
3. L

Then the prompt P may be written as:

$$(1 \to (2 \wedge 3)) \wedge (2 \to (1 \wedge 3)) \wedge (3 \to (1 \wedge 2))$$

Q5: Lemma

P is true iff 1, 2, 3 all true or all false.

Proof.

Either none, one, two, or all of the propositions 1, 2, 3 must hold:

- If all of three are true, then each conjunct is true because both the antcedents and consequents of the nested implications are true.
- If all three are false, then each conjunct is true because, for each nested implication, the antecedent is false.
- If only one is false, then the implications that do not have that proposition as their antcedents are false, because, in those implications, one of the two conjuncts in the consequent is false.
- If only two are false, then the implication having the only true proposition as antecedent is false, since its consequent is false.

□

Q5: Proof

We consider whether P is true in each case, yielding the table below.

G	H	L	1	2	3	P
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	F	T	T	F
T	F	F	F	T	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	T
F	F	T	T	F	T	F
F	F	F	T	F	F	F

There are only two scenarios where P is true: $\{G, H, L\}$ and $\{\neg G, H, \neg L\}$. Since H is the only proposition that holds in both applicable cases, H is the most likely proposition.

Q6: Statement

Elmer argues to atheist Triya that PC can be used to prove God exists.

He argues: Since, by definition of atheism, $A \to \neg G$, obviously $\neg(G \to A)$; but, $\neg(G \to A) \vdash G \wedge \neg A$, therefore $G \wedge \neg A$.

Should Triya believe Elmer's reasoning if she is presently rational, has understood argument?

Q6: Answer

No, Tryia should not believe that God exists based on Elmer's reasoning.

If $\neg(G \to A)$ is obvious, it should follow from the definition of atheism, but it doesn't. Since $A \leftrightarrow \neg G$, not just $A \to \neg G$, there are two possibilities $G \wedge \neg A$ or $\neg G \wedge A$. For $\neg(G \to A)$ to follow from the definition, it must be true in both cases. In the first case, as Elmer argues, the assertion $\neg(G \to A)$ holds. But, in the second case, the assertion does not hold (e.g., by inspection of truth tables).

Q7: Statement

Given following cards w/ number on on side and letter on other:

$\boxed{\text{I}}\quad \boxed{\text{S}}\quad \boxed{2}\quad \boxed{9}$

Which should you flip over to check the rule 'If there is a vowel on one side, then there is an even number on the other'?

Q7: Answer

We should flip over only I and 9.

Proof.

We can formalize the rule as $V \to E$. An implication is false iff the antecedent is true while, at the same time, the consequent is false:

- The S-card follows the rule because S is not a vowel, so the antecedent is false.
- The 2-card follows the rule, because 2 is an even number, so the consequent is true.
- For the I-card, we know the antecedent is true b/c I is a vowel, but we do not know whether the consequent is false.
- For the 9-card, we know the consequent is false b/c 9 is odd, but we don't know whether the antecedent is true.

Since time is money and flipping cards takes time, we should flip as few cards as possible. We can do witout flipping the S-card and the 2-card, so we should only flip the I-card and the 9-card. □

Q8: Statement

Given following cards w/ number on on side and letter on other:

$\boxed{\text{J}}$ $\boxed{\text{U}}$ $\boxed{6}$ $\boxed{7}$

Which should you flip over to check the rule 'If there is a **vowel or consonant** on one side, then there is a prime number on the other'?

Q8: Answer

You should flip none of the cards.

Proof.
We know that each card has one roman letter on one side and one digit on the other. Since all letters are either vowels or consonants, we know the antecedent holds of the 6-card. We also know that $\frac{6}{2} = 3$. Since a prime number is a positive integer that may only be divided by 1 and itself, it follows that 6 is not prime. Therefore, we know that the consequent does not hold of the 6-card. We've found a card that satisfies the antecedent but not the consequent. Therefore, we may conclude that the given cards violate the rule without flipping any of them. Since time is money, we should therefore not waste any time flipping cards when we can arrive at an answer without doing so. $\square$

Q9: Statement

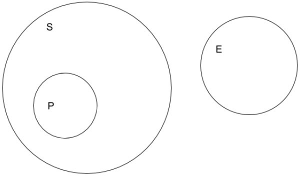
All peanut-eating animals are small.
No elephants are small.
Therefore:
No elephants are peanut-eating animals.

Does the conclusion logically follow from the premises? Justify.

Q9: Answer & Diagrammatic Proof

Yes, the conclusion logically follows from premises.

Proof.
S = Small Animals, P = Peanut-eating Animals, E = Elephants.
From premises, we have:



Since E does not intersect P, we can conclude that there are no peanunt-eating elephants. □

Q9: In First-Order Logic

FOL extends PC through addition of universal ($\forall$, 'for all') and existential ($\exists$, 'there exists') quantification, along with the ability to represent individuals. We can write the argument in FOL as:

$$\begin{array}{c} \forall x \ P(x) \to S(x) \\ \neg\exists x \ E(x) \land S(x) \\ \hline \neg\exists x \ P(x) \land E(x) \end{array}$$

PC inference rules still apply, but we need to use additional *quantifier rules* to convert quantified statements to and from propositional form.

Q9: Proof Plan in FOL

We will use proof by contradiction (AKA *reductio ad absurdum*).

This uses the the fact that $\vdash_{PC} \neg(q \wedge \neg q)$ (noncontradiction). We will write $\perp$ for $q \wedge \neg q$.

Proof by contradiction is based on modus tollens & noncontradiction:

$$\frac{\begin{array}{l} \neg \perp \\ \neg p \rightarrow \perp \end{array}}{p}$$

So, all you need to do is show $\neg p \rightarrow \perp$.

Q9: Reductio In FOL

Suppose, for contradiction, that $\exists x\ P(x) \wedge E(x)$. Let a be an arbitrary example here, hence we have $P(a) \wedge E(a)$ (existential instantiation). From first premise, we can infer $P(a) \rightarrow S(a)$ (universal instantiation). From $P(a)$, $P(a) \rightarrow S(a)$, we can infer $S(a)$ using modus ponens. But then, we have $E(a) \wedge S(a)$ and therefore $\exists x\ E(x) \wedge S(x)$ (existential generalization) which contradicts premise 2.

Q10: Statment

If the following is **either all true or all false**:

(a) $A \rightarrow B$ for 'If Alvin is happy so is Betty.'

(b) $B \rightarrow C$ for 'If Betty is happy so is Charlie is too.'

(c) $C \rightarrow D$ for 'If Charlie is happy, Darla is happy as well.'

(d) A for 'Alvin is happy.'

Does it follow deductively that 'Darla is happy'?

Q10: Answer

Yes, it follows deductively that 'Darla is happy'.

Proof.

If all given propositions are true, then we can prove Darla is happy through repeated application of modus ponens starting with propositions (a) and (d), then (b), then (c).

If all given propositions are false, we have $\neg(A \rightarrow B)$ and $\neg(B \rightarrow C)$ from premises a and b. We may rewrite these as $\neg A \wedge B$ and $\neg B \wedge C$. Then, using simplification, we can extract B and $\neg B$ from each of these intermediate results respectively.

Finally, using the explosion rule, we can infer, from B and $\neg B$, any arbitrary proposition, including D.

Since in all cases, we can prove D, it follows that we can deductively infer D from the given prompt. $\square$

Q11: Statement

In a wise-man-like scenario, we know:

- Wise men Sam, Roger know q.
- $\{q\} \vdash_{PC} r$.

Does it follow that:

1. r is true, and
2. Roger knows Sam knows r?

Q11: Answer & Proof Plan

Yes, Roger knows that Sam knows r, and, yes, it follows that r.

We use these important features of scenario in proofs:

- Both are wise men. That is, if something can be deduced about the puzzle, they will deduce it.
- The puzzle is fair, so they both have access to the same information.
- The wise men know the puzzle is fair.

We also use the general fact that if something is known, it is true.

We first prove that r follows, then we show that Roger knows Sam knows r.

Q11: Proof that r follows

We are given that both wise men know q. If a proposition is known, it must be true. Therefore, we can infer that q is true. The second assumption is that q proves r. Since we just showed that we q and since, by assumption, q proves r, it follows from the assumptions that r.

Q11: Proof that Roger knows Sam knows r

Since $\{q\} \vdash r$, $q \rightarrow r$ is a tautology ($\vdash q \rightarrow r$). This means that both Roger and Sam have all information necessary to deduce that q proves r. Since both Roger and Sam are wise men, they will deduce that $\{q\} \vdash r$.
Since Sam is a wise man, knows q and knows q proves r, Sam will deduce r and therefore know it. By the same line of argument, *mutatis mutandis*, we can conclude that Roger also knows r. Now, since Roger knows the puzzle is fair and that Sam is a wise man, he knows that Sam knows q and also that Sam knows q proves r. But from this, Roger can conclude that Sam also knows r since Same is a wise man and will therefore make the inference.

Q12: Statement

Monty hall problem w/ 47 doors:

- You pick 1 of 47 doors, behind one of which there is a prize.
- Monty reveals 1 of **remaining 46** that does not have a prize.
- You get to stay on same door, or switch to a new one.

In this scenario, should you still switch just as in the 3-door version?

Q12: Answer

You should still switch.

Proof.
In a hold policy, the odds of losing are $46/47$ since there are 46 llamas and 1 million dollar prize. In a switch policy, there are two cases:

- 1/47 chance of initially choosing the million, leading to a certain loss.
- 46/47 chance of initially choosing llama, followed by a 44/45 chance of switching to the million.

Thus the probability of losing with a switch policy is

$$\frac{1}{47} + \frac{46}{47}\frac{44}{45} = \frac{2069}{2115}$$

Since $46/47 = 2070/2115 > 2069/2115$, the swicting policy is slightly more likely to win. $\square$

Q13: The Bi-Pay Auction

Is it rational to participate in an auction of the following form? Justify your answer.

- Participants bid for a pot of S\$ (in class, $S = 60$).
- The winner of a round of bidding pledges to pay R dollars, where R is the number of rounds. So, on the first round, the winner pledges to pay 1\$, on the second round 2\$ etc.
- If the auction ends on the Rth round, the winner takes the pot but must pay the amount pledged (therefore the winner gets a payoff of $S - R$\$) and the runner up (winner of previous round) must also pay the amount they had pledged in the previous round (so they get a 'payoff' of $1 - R$\$).

Q13: Answer?

Open question! For further reading, see:
`http://kryten.mm.rpi.edu/JJ_NSG_SB_bounded_rationality_031214.pdf`
Marks off if:

- Don't know the scenario
- Do not touch on the runner-up's incentive to bid again
- Provide no justification for response
- Say nothing